

# UNIQUENESS OF SELF-SHRINKERS TO THE DEGREE-ONE CURVATURE FLOW WITH A TANGENT CONE AT INFINITY

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ABSTRACT. Given a smooth, symmetric, homogeneous of degree one function  $f = f(\lambda_1, \dots, \lambda_n)$  satisfying  $\partial_i f > 0$  for all  $i = 1, \dots, n$ , and an oriented, properly embedded smooth cone  $\mathcal{C}^n$  in  $\mathbb{R}^{n+1}$ , we show that under some suitable conditions on  $f$  and the covariant derivatives of the second fundamental form of  $\mathcal{C}$ , there is at most one  $f$  self-shrinker (i.e. an oriented hypersurface  $\Sigma^n$  in  $\mathbb{R}^{n+1}$  for which  $f(\kappa_1, \dots, \kappa_n) + \frac{1}{2}X \cdot N = 0$  holds, where  $X$  is the position vector,  $N$  is the unit normal vector, and  $\kappa_1, \dots, \kappa_n$  are principal curvatures of  $\Sigma$ ) that is asymptotic to the given cone  $\mathcal{C}$  at infinity.

## 1. INTRODUCTION

Let  $\mathcal{C}^n$  be an oriented, properly embedded smooth cone (excluding the vertex  $O$ ) in  $\mathbb{R}^{n+1}$ . Suppose that  $\Sigma^n$  is an oriented, properly embedded smooth hypersurface in  $\mathbb{R}^{n+1}$  which satisfies

$$H + \frac{1}{2}X \cdot N = 0 \quad \forall X \in \Sigma$$

$$\varrho \Sigma \xrightarrow{C_{loc}^\infty} \mathcal{C} \quad \text{as } \varrho \searrow 0$$

where  $X$  is the position vector,  $N$  is the unit normal vector and  $H = -\nabla_\Sigma \cdot N$  is the mean curvature of  $\Sigma$ . Then  $\Sigma$  is called a self-shrinker to the mean curvature flow (MCF, an one-parameter family of hypersurfaces for which  $\partial_t X_t^\perp = HN$  holds) which is smoothly asymptotic to the cone  $\mathcal{C}$  at infinity. It follows that the rescaled family of hypersurfaces  $\{\Sigma_t = \sqrt{-t}\Sigma\}$  forms a mean curvature flow starting from  $\Sigma$  (when  $t = -1$ ) and converging locally smoothly to  $\mathcal{C}$  as  $t \nearrow 0$ . Wang in [W] proves the uniqueness of such self-shrinkers by showing the following: suppose  $\tilde{\Sigma}$  is also a self-shrinker which is asymptotic to the same cone  $\mathcal{C}$  at infinity, then outside a large ball  $B_R^{n+1}$ , each  $\tilde{\Sigma}_t = \sqrt{-t}\tilde{\Sigma}$  can be regarded as a normal graph of  $\mathbf{h}_t$  defined on  $\Sigma_t \setminus \bar{B}_R$  for some  $R > 0$ ; moreover, given  $\varepsilon > 0$  and choose  $R$  sufficiently large, there holds

$$\begin{aligned} \left| \partial_t \mathbf{h} - \Delta_{\Sigma_t} \mathbf{h} \right| &\leq \varepsilon (|\nabla_{\Sigma_t} \mathbf{h}| + |\mathbf{h}|) \\ \mathbf{h} \Big|_{t=0} &= 0 \end{aligned}$$

Then using the idea of [ESS], Wang derives a Carleman's inequality for the heat operator on  $\{\Sigma_t\}$ , applies it to the localization of  $\mathbf{h}$ , and then uses the unique continuation principle (see [EF], for instance) to conclude that  $\mathbf{h} = 0$ .

On the other hand, Andrews in [A] consider the motion of hypersurfaces in  $\mathbb{R}^{n+1}$  moved by some degree one curvature. More precisely, given a smooth, symmetric and homogeneous of degree-one function  $f = f(\lambda_1, \dots, \lambda_n)$  which satisfies

$$\partial_i f > 0, \quad \forall i = 1, \dots, n$$

consider the following evolution of hypersurfaces:

$$\partial_t X_t^\perp = f(\kappa_1, \dots, \kappa_n) N$$

where  $\kappa_1, \dots, \kappa_n$  are the principal curvatures of the evolving hypersurface. For example, if we take the curvature function to be  $f(\lambda_1, \dots, \lambda_n) = \lambda_1 + \dots + \lambda_n$ , then it corresponds to the mean curvature flow. We call an oriented  $C^2$  hypersurface  $\Sigma^n$  in  $\mathbb{R}^{n+1}$  to be a “ $f$  self-shrinker” to the above “ $f$  curvature flow” provided that

$$f(\kappa_1, \dots, \kappa_n) + \frac{1}{2} X \cdot N = 0$$

holds on  $\Sigma$ . Examples of  $f$  self-shrinker can be found in [G]. Just like the MCF, the rescaled family of “ $f$  self-shrinkers” is a self-similar solution to the  $f$  curvature flow; that is, the rescaled family of hypersurfaces  $\{\Sigma_t = \sqrt{-t}\Sigma\}_{t < 0}$  forms a “ $f$  curvature flow”. In the case when  $\Sigma$  is smoothly asymptotic to the cone  $\mathcal{C}$  at infinity, the rescaled flow  $\{\Sigma_t\}_{t < 0}$  converges locally smoothly to  $\mathcal{C}$  as  $t \nearrow 0$ .

This paper is an extension of the uniqueness result of [W] to the class of  $f$  self-shrinkers with a tangent cone at infinity. Based on Wang’s idea of proving the uniqueness, we need to have some additional treatments to the nonlinearity of  $f$  (which is not a concern in Wang’s case because the curvature function there is linear) in order to generalize the result. The crucial step is to derive Carleman’s inequality for the associated parabolic operator to the “ $f$  curvature flow” under some assumptions on the nonlinearity of  $f$ , the uniform positivity of  $\partial_i f$  and some curvature bounds of  $\mathcal{C}$  (see Proposition 4.11). For this part, we are motivated by the work of Nguyen in [N] as well as Wu and Zhang in [WZ] for deriving Carleman’s inequality for parabolic operator with variable coefficients (see Remark 4.10).

In order to state our main results, Theorem 2.5, we have to first introduce some notations and definitions regarding the  $f$  self-shrinkers, the tangent cone of a hypersurface at infinity, and also some basic assumptions on the curvature function  $f$ . We put all of these in Section 2.

In Section 3, we essentially follow the idea of [W]: if  $\Sigma^n$  and  $\tilde{\Sigma}^n$  are  $f$  self-shrinkers which are asymptotic to the given cone  $\mathcal{C}^n$  at infinity, then outside a large ball  $B_R^{n+1}$ ,  $\tilde{\Sigma}_t = \sqrt{-t}\tilde{\Sigma}$  can be regarded as a normal graph of  $h_t$  defined on  $\Sigma_t \setminus \bar{B}_R$  (see Lemma 3.1), which satisfies some parabolic equation and vanishes at time 0 (see Proposition 3.7). We also give some estimates on the coefficients of the parabolic equation (see Proposition 3.8).

In Section 4, we follow the idea of [ESS] for treating the backward uniqueness of the heat equation (which is also used in [W] to deal with the uniqueness of self-shrinkers to the MCF) to show that the function  $h_t$  vanishes outside a large ball  $B_R$ . We would first apply the mean value inequality for parabolic equations and a local type of Carleman’s inequalities to show the exponential decay of  $h_t$  to 0 as  $t \nearrow 0$  as in [N] (see Proposition 4.7). Then we are devoted to derive a global type of Carleman’s inequalities (based on the estimates of the coefficients of the parabolic equation derived in Section 3, see Proposition 4.11) and use it to show that  $h_t$  vanishes outside a ball  $B_R$ ; in other words, the two shrinkers coincide outside a ball  $B_R$ . In the end, we use the unique continuation principle to characterize the overlap region of  $\Sigma$  and  $\tilde{\Sigma}$ .

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## 2. ASSUMPTIONS AND MAIN RESULTS

**Definition 2.1** (A regular cone). Let  $\mathcal{C}^n$  be an oriented and properly embedded smooth cone (excluding the vertex  $O$ ) in  $\mathbb{R}^{n+1}$ ; that is,  $\mathcal{C}$  is an oriented and properly embedded hypersurface in  $\mathbb{R}^{n+1}$  satisfying  $\varrho\mathcal{C} = \mathcal{C}$  for all  $\varrho \in \mathbb{R}_+$ , and we assume that  $O \notin \mathcal{C}$ .

We then define what it means for a hypersurface to be asymptotic to the cone  $\mathcal{C}$  at infinity.

**Definition 2.2** (Tangent cone at infinity). A  $C^k$  hypersurface  $\Sigma^n$  in  $\mathbb{R}^{n+1}$  (with  $k \in \mathbb{N}$ ) is said to be  $C^k$  asymptotic to  $\mathcal{C}$  at infinity provided that  $\varrho\Sigma \xrightarrow{C_{\text{loc}}^k} \mathcal{C}$  as  $\varrho \searrow 0$ . In this case,  $\mathcal{C}$  is called the tangent cone of  $\Sigma$  at infinity.

For a given  $C^2$  oriented hypersurface  $\Sigma^n$  in  $\mathbb{R}^{n+1}$ , its Weingarten map  $A^\#$  sends tangent vectors to tangent vectors in such a way that

$$A^\#(V) = -D_V N$$

for any tangent vector field  $V$  on  $\Sigma$ , where  $N$  is the unit-normal of  $\Sigma$ . The second fundamental form  $A$  is defined to be a 2 tensor on  $\Sigma$  so that

$$A(V, W) = A^\#(V) \cdot W$$

for any tangent vector fields  $V$  and  $W$  on  $\Sigma$ . The components of  $A^\#$  and  $A$  with respect to a given local frame  $\{e_1, \dots, e_n\}$  of the tangent bundle of  $\Sigma$  are defined by

$$A^\#(e_i) = A_i^j e_j, \quad A(e_i, e_j) = A_{ij}$$

For simplicity,  $A^\#$  and  $A$  are usually denoted by their components:  $A^\# \sim A_i^j$  and  $A \sim A_{ij}$ . Note that  $A^\#$  is a self-adjoint operator with respect to the induced metric of the hypersurface (or equivalently,  $A$  is a symmetric 2 tensor), so  $A^\#$  is diagonalizable. The eigenvectors of  $A^\#$  are called principal vectors and the associated eigenvalues are called principal curvatures, which we denote by  $\kappa_1, \dots, \kappa_n$ . The mean curvature is defined to be  $H = \text{tr}(A^\#) = \kappa_1 + \dots + \kappa_n$ , which is a linear, symmetric and homogeneous of degree-one function of the shape operator (or the principal curvatures). In this paper, we consider a generic degree-one curvature.

**Definition 2.3** (The degree-one curvature function). Let  $F = F(S)$  be a conjugation-invariant, homogeneous of degree-one function whose domain  $\Omega$  (in the space of  $n \times n$  matrices) containing a neighborhood of the set consisting of all the values of shape operator  $A_\mathcal{C}^\#$  of  $\mathcal{C}$ ; besides,  $F$  can be written as a  $C^3$  function composed with the elementary symmetric functions  $\mathcal{E}_1, \dots, \mathcal{E}_n$  (for instance,  $\mathcal{E}_1 = \text{tr}$  and  $\mathcal{E}_n = \det$ ) and  $\frac{\partial F}{\partial S_i^j} > 0$  (i.e.  $\frac{\partial F}{\partial S_i^j}$  is a positive matrix). In particular, we require the curvature function  $F$  to be defined and  $C^3$  on the curvature of  $\mathcal{C}$ .

Note that by the conjugation-invariant and homogeneous property of  $F$ , we may assume that  $\Omega$  is closed under conjugation and homothety; that is, if  $S \in \Omega$ , then so are  $RSR^{-1}$  and  $\varrho S$  for any invertible  $n \times n$  matrix  $R$  and positive number  $\varrho$ .

Also, by the condition that  $F$  can be written as a  $C^3$  function composed with the elementary symmetric functions, it induces a symmetric, homogeneous of degree-one function  $f$  so that

$$F(S) = f(\lambda_1, \dots, \lambda_n)$$

whenever  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of the matrix  $S$ . The function  $f$  is defined and  $C^3$  on an open set  $\mathcal{U}$  (in  $\mathbb{R}^n$ ) containing a neighborhood of the set consisting of all the values of the principal curvature vector  $(\kappa_1^{\mathcal{C}}, \dots, \kappa_n^{\mathcal{C}})$  of  $\mathcal{C}$ . Likewise, we may assume that the domain  $\mathcal{U}$  is closed under permutation and homothety.

In fact, at a diagonal matrix  $S = \text{diag}(\lambda_1, \dots, \lambda_n)$ , there holds (see [A]):

$$(2.1) \quad \frac{\partial F}{\partial S_i^j}(S) = \partial_i f(\lambda_1, \dots, \lambda_n) \delta_{ij}$$

$$(2.2) \quad \frac{\partial^2 F}{\partial S_i^j \partial S_i^l}(S) = \partial_{ii}^2 f(\lambda_1, \dots, \lambda_n) \delta_{ij} \delta_{il}$$

$$(2.3) \quad \frac{\partial^2 F}{\partial S_i^j \partial S_k^l}(S) = \partial_{ik}^2 f(\lambda_1, \dots, \lambda_n) \delta_{ij} \delta_{kl} + \frac{\partial_i f - \partial_k f}{\lambda_i - \lambda_k} \delta_{il} \delta_{kj} \quad \text{if } i \neq k$$

Since  $F$  is well-defined on conjugacy classes, (2.1), (2.2), (2.3) can be applied to any diagonalizable matrix in  $\Omega$ . For example, by (2.1), we have

$$\frac{\partial F}{\partial S_i^j}(A_{\mathcal{C}}^{\#}) \sim \partial_i f(\kappa_1^{\mathcal{C}}, \dots, \kappa_n^{\mathcal{C}}) \delta_{ij}$$

where  $A_{\mathcal{C}}^{\#} \sim \kappa_{\mathcal{C}}^i \delta_{ij}$  are the shape operator and principal curvatures of  $\mathcal{C}$ , respectively. Besides, by the condition that  $\frac{\partial F}{\partial S_i^j} > 0$  on  $\Omega$ , we may assume that  $\partial_i f > 0 \quad \forall i = 1, \dots, n$  on  $\mathcal{U}$ .

Let  $U$  be an open neighborhood of the set consisting of the all the shape operator  $A_{\mathcal{C}}^{\#}$  of  $\mathcal{C}$  at each  $X_{\mathcal{C}} \in \mathcal{C} \cap (B_3 \setminus \bar{B}_{\frac{1}{3}})$  in  $\Omega$ . Note that we may assume that  $U$  is closed under conjugation and that  $\frac{\partial F}{\partial S_i^j}$  is uniformly positive on  $U$ ; that is, there exist a constant  $\lambda \in (0, 1]$  so that

$$(2.4) \quad \lambda \delta_j^i \leq \frac{\partial F}{\partial S_i^j} \leq \frac{1}{\lambda} \delta_j^i$$

Also, we have

$$(2.5) \quad \begin{aligned} \varkappa &\equiv \sup_{X_{\mathcal{C}} \in \mathcal{C} \cap (B_3 \setminus \bar{B}_{\frac{1}{3}})} \left| \nabla_{\mathcal{C}} \left( \frac{\partial F}{\partial S_i^j}(A_{\mathcal{C}}^{\#}) \right) \right| \\ &= \sup_{X_{\mathcal{C}} \in \mathcal{C} \cap (B_3 \setminus \bar{B}_{\frac{1}{3}})} \left| \sum_{k,l} \frac{\partial^2 F}{\partial S_i^j \partial S_k^l}(A_{\mathcal{C}}^{\#}) \left( \nabla_{\mathcal{C}} A_{\mathcal{C}}^{\#} \right)_k^l \right| \\ &\leq C(n, \mathcal{C}, \|F\|_{C^2(U)}) \end{aligned}$$

where  $A_{\mathcal{C}}^{\#}$  and  $\nabla_{\mathcal{C}} A_{\mathcal{C}}^{\#}$  are the shape operator of  $\mathcal{C}$  and its covariant derivative at  $X_{\mathcal{C}}$ , respectively;  $B_{\varrho} = B_{\varrho}^{n+1}$  is the ball of radius  $\varrho$  in  $\mathbb{R}^{n+1}$ . We would give a more precise estimate of  $\varkappa$  in Section 5 (see (4.95)) in the case when  $\mathcal{C}$  is rotationally symmetric.

Now let's define the  $F$  self-shrinker (or  $f$  self-shrinker):

**Definition 2.4** ( $F$  self-shrinker /  $f$  self-shrinker). An oriented  $C^2$  hypersurface  $\Sigma^n$  in  $\mathbb{R}^{n+1}$  is called a  $F$  self-shrinker (or  $f$  self-shrinker) provided that  $F$  is defined on the shape operator  $A^{\#}$  of  $\Sigma$  (i.e.  $A^{\#} \in \Omega$ ) and there holds

$$F(A^{\#}) + \frac{1}{2}X \cdot N = 0$$

where  $X$  is the position vector,  $N$  is the unit-normal, and  $A^{\#}$  is the shape operator of  $\Sigma$ ; or equivalently,  $f$  is defined on the principal curvatures of  $\Sigma$  (i.e.  $(\kappa_1, \dots, \kappa_n) \in \mathcal{U}$ ) and there holds

$$f(\kappa_1, \dots, \kappa_n) + \frac{1}{2}X \cdot N = 0$$

where  $\kappa_1, \dots, \kappa_n$  are the principal curvatures of  $\Sigma$ .

Note that the rescaled family of  $F$  self-shrinkers forms a self-similar solution to the  $F$  curvature flow. More precisely, the one-parameter family  $\{\Sigma_t = \sqrt{-t}\Sigma\}_{-1 \leq t < 0}$  is a motion of a hypersurface moved by  $F$  curvature vector. That is,

$$\partial_t X_t^{\perp} = F(A^{\#})N$$

where  $\partial_t X_t^{\perp}$  is the normal projection of  $\partial_t X_t$ . Besides, for each time-slice  $\Sigma_t = \sqrt{-t}\Sigma$ , there holds

$$F(A^{\#}) + \frac{X \cdot N}{2(-t)} = 0$$

We would prove the following uniqueness result for  $F$  self-shrinkers with a tangent cone in Section 4.

**Theorem 2.5** (Uniqueness of self-shrinkers with a conical end). *Assume that  $\varkappa \leq 6^{-4}\lambda^3$  (see (2.4), (2.5)). Then for any properly embedded  $F$  self-shrinkers  $\Sigma^n$  and  $\tilde{\Sigma}^n$  which are  $C^5$  asymptotic to the cone  $\mathcal{C}$  at infinity, there exists  $R = R(\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \|F\|_{C^3(U)}, \lambda, \varkappa) \geq 1$  so that  $\Sigma \setminus B_R = \tilde{\Sigma} \setminus B_R$ . Moreover, let*

$$\Sigma^0 = \left\{ X \in \Sigma \cap \tilde{\Sigma} \mid \Sigma \text{ coincides with } \tilde{\Sigma} \text{ in a neighborhood of } X \right\}$$

then  $\Sigma^0$  is a nonempty hypersurface, which satisfies  $\partial\Sigma^0 \subseteq (\partial\Sigma \cup \partial\tilde{\Sigma})$ .

*Remark 2.6.* In the case of [W],  $F = \mathcal{E}_1$  (or equivalently,  $f(\lambda_1, \dots, \lambda_n) = \lambda_1 + \dots + \lambda_n$ ) is a linear function, so (by (2.5), (2.2), (2.3))  $\varkappa \equiv 0$  and the hypothesis of Theorem 2.5 is trivially satisfied. On the other hand, consider

$$F = \mathcal{E}_1 \pm \epsilon \frac{\mathcal{E}_n}{\mathcal{E}_{n-1}}$$

or equivalently,

$$f(\lambda_1, \dots, \lambda_n) = (\lambda_1 + \dots + \lambda_n) \pm \epsilon \frac{\prod_{i=1}^n \lambda_i}{\sum_{i=1}^n \left( \prod_{j \neq i} \lambda_j \right)}$$

and take  $\mathcal{C}$  to be a rotationally symmetric cone. Then by Theorem 2.5 and (4.95) in the Section 5, the uniqueness holds when  $0 < \epsilon \ll 1$ .

### 3. DEVIATION BETWEEN TWO $F$ SELF-SHRINKERS WITH THE SAME ASYMPTOTIC BEHAVIOR AT INFINITY

Let  $\Sigma^n$  be a properly embedded  $F$  self-shrinker (in Definition 2.4) which is  $C^5$  asymptotic to the cone  $\mathcal{C}$  at infinity.

By Definition 2.2,  $\varrho\Sigma$  can be arbitrary  $C^5$  close to  $\mathcal{C}$  on any fixed bounded set of  $\mathbb{R}^{n+1}$  which is away from the origin (e.g. on  $B_2 \setminus \bar{B}_{\frac{1}{2}}$ ) as long as  $\varrho$  is sufficiently small. Below we would like to use this condition to show that any “rescaled”  $C^5$  quantities of  $\Sigma \setminus \bar{B}_R$  can be estimated by that of  $\mathcal{C}$  (if  $R$  is sufficiently large).

First, choose  $R \gg 1$  (depending on  $\Sigma, \mathcal{C}$ ) so that outside a compact set,  $\Sigma$  is a normal graph over  $\mathcal{C} \setminus \bar{B}_R$ , say  $X = \Psi(X_C) = X_C + \psi N_C$ , where  $X_C$  is the position vector of  $\mathcal{C}$ ,  $N_C$  is the unit-normal of  $\mathcal{C}$  at  $X_C$  and  $\psi$  is a real-valued function of  $X_C$ . Consequently, it’s natural to define the “normal projecton”  $\Pi$  to be the inverse map of  $\Psi$ , which sends  $X \in \Sigma$  to  $X_C \in \mathcal{C}$ . Also, by the rescaling argument, we may assume that

$$\mathcal{H}^n(\Sigma \cap (B_{2r} \setminus \bar{B}_r)) \leq C(n, \mathcal{C}) r^n$$

for all  $r \geq R$  (i.e.  $\Sigma$  has polynomial volume growth).

On the other hand, for each fixed  $\hat{X}_C \in \mathcal{C} \setminus \bar{B}_R$ , we have  $|\hat{X}_C|^{-1} \hat{X}_C \in \mathcal{C}$ . So near  $|\hat{X}_C|^{-1} \hat{X}_C$ ,  $\mathcal{C}$  is locally a graph over its tangent hyperplane at  $|\hat{X}_C|^{-1} \hat{X}_C$ . By Definition 2.2,  $|\hat{X}_C|^{-1} \Sigma$  is  $C^5$  close to  $\mathcal{C}$ , so it must also be a local graph over  $T_{|\hat{X}_C|^{-1} \hat{X}_C} \mathcal{C}$ , and the graph must be  $C^5$  close to the corresponding graph of  $\mathcal{C}$ .

Furthermore, by [L], there exists a uniform radius  $\rho \in (0, \frac{1}{8}]$  (depending on the dimension  $n$ , the volume of  $\mathcal{C} \cap (B_3 \setminus \bar{B}_{\frac{1}{3}})$  and the  $C^3$  bound of the curvature of  $\mathcal{C} \cap (B_3 \setminus \bar{B}_{\frac{1}{3}})$ ) so that near  $|\hat{X}_C|^{-1} \hat{X}_C$ , the graph of  $\mathcal{C}$  and the graph of  $|\hat{X}_C|^{-1} \Sigma$  are defined on  $B_\rho^n \subset T_{|\hat{X}_C|^{-1} \hat{X}_C} \mathcal{C}$ . We may also assume that the  $C^1$  norm of the local graph of  $\mathcal{C}$  on  $B_\rho^n \subset T_{|\hat{X}_C|^{-1} \hat{X}_C} \mathcal{C}$  is small (by choosing  $\rho$  small).

After undoing the rescaling (from  $|\hat{X}_C|^{-1} \Sigma$  to  $\Sigma$ ), the above translates into the following: there exists  $R = R(\Sigma, \mathcal{C}) \geq 1$  so that near each  $\hat{X}_C \in \mathcal{C} \setminus \bar{B}_R$ ,  $\mathcal{C}$  and  $\Sigma$  can be respectively parametrized by

$$X_C = X_C(x) \equiv \hat{X}_C + (x, \mathbf{w}(x))$$

$$X = X(x) \equiv \hat{X}_C + (x, \mathbf{u}(x))$$

for  $x = (x_1, \dots, x_n) \in B_{\rho|\hat{X}_C|}^n \subset T_{\hat{X}_C} \mathcal{C}$ , such that  $\mathbf{w}(0) = 0$ ,  $\partial_x \mathbf{w}(0) = 0$  and

$$(3.1) \quad |\hat{X}_C|^{-1} \|\mathbf{w}\|_{L^\infty(B_{\rho|\hat{X}_C|}^n)} + \|\partial_x \mathbf{w}\|_{L^\infty(B_{\rho|\hat{X}_C|}^n)} \leq \frac{1}{16}$$

$$(3.2) \quad |\hat{X}_C| \|\partial_x^2 \mathbf{w}\|_{L^\infty(B_{\rho|\hat{X}_C|}^n)} + \dots + |\hat{X}_C|^4 \|\partial_x^5 \mathbf{w}\|_{L^\infty(B_{\rho|\hat{X}_C|}^n)} \leq C(n, \mathcal{C})$$

$$\begin{aligned}
(3.3) \quad & \{ |\hat{X}_C|^{-1} \| \mathbf{u} - \mathbf{w} \|_{L^\infty(B_{\rho|\hat{X}_C|}^n)} + \| \partial_x \mathbf{u} - \partial_x \mathbf{w} \|_{L^\infty(B_{\rho|\hat{X}_C|}^n)} \\
& + |\hat{X}_C| \| \partial_x^2 \mathbf{u} - \partial_x^2 \mathbf{w} \|_{L^\infty(B_{\rho|\hat{X}_C|}^n)} + \cdots \\
& + |\hat{X}_C|^4 \| \partial_x^5 \mathbf{u} - \partial_x^5 \mathbf{w} \|_{L^\infty(B_{\rho|\hat{X}_C|}^n)} \} \leq \frac{1}{16}
\end{aligned}$$

where we assume the unit-normal of  $\mathcal{C}$  at  $\hat{X}_C$  to be  $(0, 1)$  for ease of notation (and hence  $\Pi(X(0)) = \hat{X}_C$ ). Note that (3.1) is the rescale of the smallness of the  $C^1$  norm of the local graph of  $\mathcal{C}$ , and (3.3) is the rescale of the small  $C^5$  difference between the local graphs of  $\mathcal{C}$  and  $|\hat{X}_C|^{-1}\Sigma$ .

By Definition 2.2 and the rescaling argument, the same thing holds for each rescaled hypersurface  $\Sigma_t = \sqrt{-t}\Sigma$ ,  $t \in [-1, 0)$  as well. That is, outside a compact set,  $\Sigma_t$  is a normal graph over  $\mathcal{C} \setminus \bar{B}_R$  (with  $R \gg 1$  depending on  $\Sigma, \mathcal{C}$ ); besides, near each  $\hat{X}_C \in \mathcal{C} \setminus \bar{B}_R$ ,  $\Sigma_t$  is a graph over  $T_{|\hat{X}_C|^{-1}\hat{X}_C}\mathcal{C}$  and it can be parametrized by

$$X_t(x) = X(x, t) \equiv \hat{X}_C + (x, \mathbf{u}_t(x)) = \hat{X}_C + (x, \mathbf{u}(x, t))$$

which satisfies

$$\begin{aligned}
(3.4) \quad & \{ |\hat{X}_C|^{-1} \| \mathbf{u}(\cdot, t) - \mathbf{w} \|_{L^\infty(B_{\rho|\hat{X}_C|}^n)} + \| \partial_x \mathbf{u}(\cdot, t) - \partial_x \mathbf{w} \|_{L^\infty(B_{\rho|\hat{X}_C|}^n)} \\
& + |\hat{X}_C| \| \partial_x^2 \mathbf{u}(\cdot, t) - \partial_x^2 \mathbf{w} \|_{L^\infty(B_{\rho|\hat{X}_C|}^n)} + \cdots \\
& + |\hat{X}_C|^4 \| \partial_x^5 \mathbf{u}(\cdot, t) - \partial_x^5 \mathbf{w} \|_{L^\infty(B_{\rho|\hat{X}_C|}^n)} \} \leq \frac{1}{16}
\end{aligned}$$

We call  $t \mapsto X(x, t) = \hat{X}_C + (x, \mathbf{u}(x, t))$  to be the “vertical parametrization” of the flow  $\{\Sigma_t\}_{-1 \leq t < 0}$ . By (3.1), (3.4) and  $0 < \rho \leq \frac{1}{8}$ , we have

$$\frac{3}{4}|\hat{X}_C| \leq |X(x, t)| = |\hat{X}_C + (x, \mathbf{u}(x, t))| \leq \frac{5}{4}|\hat{X}_C|$$

for  $x \in B_{\rho|\hat{X}_C|}^n \subset T_{\hat{X}_C}\mathcal{C}$ ,  $t \in [-1, 0)$ ; that is,  $|X|$  is comparable with  $|\hat{X}_C|$ . Note that we still have the polynomial volume growth for each  $\Sigma_t$ :

$$(3.5) \quad \mathcal{H}^n(\Sigma_t \cap (B_{2r} \setminus \bar{B}_r)) \leq C(n, \mathcal{C})r^n$$

for all  $r \geq R$ .

On the other hand, we could use the  $F$  self-shrinker condition to improve (3.4). To see this, observe that under the conditions of being a  $F$  self-shrinker and having a tangent cone  $\mathcal{C}$  at infinity, the rescaled flow  $\{\Sigma_t = \sqrt{-t}\Sigma\}_{-1 \leq t < 0}$  moves by  $F$  curvature vector and converges (in the locally  $C^5$  sense) to the cone  $\mathcal{C}$  as  $t \nearrow 0$ . In other words, we could define a  $F$  curvature flow  $\{\Sigma_t\}_{-1 \leq t \leq 0}$  with  $\Sigma_t = \sqrt{-t}\Sigma$  for  $t \in [-1, 0)$  and  $\Sigma_0 = \mathcal{C}$ , which is continuous upto  $t = 0$  (in the locally  $C^5$  sense). Besides, near each  $\hat{X}_C \in \mathcal{C} \setminus \bar{B}_R$  (with  $R \gg 1$  depending on  $\Sigma, \mathcal{C}$ ), we have the vertical parametrization of the flow (as above) for  $t \in [-1, 0]$ . by Definition 2.4, the evolution of  $u_t$  satisfies

$$(3.6) \quad \partial_t \mathbf{u} = \sqrt{1 + |\partial_x \mathbf{u}|^2} F \left( A_t^j(x, t) \right)$$

for  $(x_1, \dots, x_n) \in B_{\rho|\hat{X}_C|}^n \subset T_{\hat{X}_C}\mathcal{C}$ ,  $-1 \leq t < 0$ , and

$$(3.7) \quad \mathbf{u}(\cdot, t) \xrightarrow{C^5} w \quad \text{on } B_{\rho|\hat{X}_C|}^n \quad \text{as } t \nearrow 0$$

where the shape operator  $A_t^\#(x) \sim A_i^j(x, t)$  of  $\Sigma_t$  (with respect to the local coordinate frame  $\{\partial_1 X_t, \dots, \partial_n X_t\}$ ) is equal to

$$(3.8) \quad A_i^j(x, t) = \partial_i \left( \frac{\partial_j \mathbf{u}(x, t)}{\sqrt{1 + |\partial_x \mathbf{u}|^2}} \right)$$

It follows (by (3.6), (3.4), (3.1), (3.2), (3.8)) that

$$\begin{aligned} |\partial_t \mathbf{u}| &= |\hat{X}_C|^{-1} \sqrt{1 + |\partial_x \mathbf{u}|^2} \left| F \left( |\hat{X}_C| A_i^j(x, t) \right) \right| \\ &\leq |\hat{X}_C|^{-1} \left( 1 + \|\partial_x \mathbf{u}_t\|_{L^\infty(B_{\rho|\hat{X}_C|}^n)} \right) \|F\|_{L^\infty(U)} \end{aligned}$$

in which we use the homogeneity of  $F$ .

Similarly, by differentiating (3.6) and using the homogeneity of  $F$  (and its derivatives), we get

$$\begin{aligned} (3.9) \quad &\{|\hat{X}_C| \|\partial_t \mathbf{u}(\cdot, t)\|_{L^\infty(B_{\rho|\hat{X}_C|}^n)} + |\hat{X}_C|^2 \|\partial_t \partial_x \mathbf{u}(\cdot, t)\|_{L^\infty(B_{\rho|\hat{X}_C|}^n)} \\ &\quad + |\hat{X}_C|^3 \|\partial_t \partial_x^2 \mathbf{u}(\cdot, t)\|_{L^\infty(B_{\rho|\hat{X}_C|}^n)} \\ &\quad + |\hat{X}_C|^4 \|\partial_t \partial_x^3 \mathbf{u}(\cdot, t)\|_{L^\infty(B_{\rho|\hat{X}_C|}^n)}\} \leq C(n, \mathcal{C}, \|F\|_{C^3(U)}) \end{aligned}$$

which implies (by (3.9) and (3.6))

$$|\mathbf{u}(\cdot, t) - w| \leq \int_t^0 |\partial_t \mathbf{u}(\cdot, \tau)| d\tau \leq C(n, \mathcal{C}, \|F\|_{C^3(U)}) |\hat{X}_C|^{-1}(-t)$$

Likewise, integrate the estimates for derivatives in (3.9) to get:  $\forall t \in [-1, 0]$

$$\begin{aligned} (3.10) \quad &\{|\hat{X}_C| \|\mathbf{u}(\cdot, t) - \mathbf{w}\|_{L^\infty(B_{\rho|\hat{X}_C|}^n)} + |\hat{X}_C|^2 \|\partial_x \mathbf{u}(\cdot, t) - \partial_x \mathbf{w}\|_{L^\infty(B_{\rho|\hat{X}_C|}^n)} \\ &\quad + |\hat{X}_C|^3 \|\partial_x^2 \mathbf{u}(\cdot, t) - \partial_x^2 \mathbf{w}\|_{L^\infty(B_{\rho|\hat{X}_C|}^n)} \\ &\quad + |\hat{X}_C|^4 \|\partial_x^3 \mathbf{u}(\cdot, t) - \partial_x^3 \mathbf{w}\|_{L^\infty(B_{\rho|\hat{X}_C|}^n)}\} \leq C(n, \mathcal{C}, \|F\|_{C^3(U)})(-t) \end{aligned}$$

which is the improvement of (3.4) by using the equation (3.6).

In view of the pull-back metric

$$g_{ij}(x, t) = \delta_{ij} + \partial_i u(x, t) \partial_j u(x, t)$$

and the associated Christoffel symbols

$$(3.11) \quad \Gamma_{ij}^k(x, t) = \frac{\partial_k \mathbf{u}(x, t) \partial_{ij}^2 \mathbf{u}(x, t)}{1 + |\partial_x \mathbf{u}(x, t)|^2}$$

together with (3.8), (3.10), the comparability of  $|X|$  and  $|\hat{X}_C|$ , (2.4), (2.5) and the continuity and homogeneity of  $F$  (and its derivatives), there exists  $R \geq 1$  (depending on  $\Sigma, \mathcal{C}, U, \|F\|_{C^3(U)}, \lambda, \varkappa$ ) such that for  $X_t \in \Sigma_t \setminus \bar{B}_R$ , the following hold:

$$(3.12) \quad |X_t| A_t^\# \in U$$



$$(3.13) \quad \frac{\lambda}{2} \delta_j^i \leq \frac{\partial F}{\partial S_i^j} (A_t^\#) = \frac{\partial F}{\partial S_i^j} (|X_t| A_t^\#) \leq \frac{2}{\lambda} \delta_j^i$$

$$(3.14) \quad |X_t| \left| \sum_{k,l} \frac{\partial^2 F}{\partial S_i^j \partial S_k^l} (A_t^\#) \left( \nabla_{\Sigma_t} A_t^\# \right)_k^l \right| = \left| \sum_{k,l} \frac{\partial^2 F}{\partial S_i^j \partial S_k^l} (|X_t| A_t^\#) \cdot (|X_t|^2 \nabla_{\Sigma_t} A_t^\#)_k^l \right| \leq 2\kappa$$

$$(3.15) \quad |X_t| |A_t^\#| + |X_t|^2 |\nabla_{\Sigma_t} A_t^\#| + |X_t|^3 |\nabla_{\Sigma_t}^2 A_t^\#| \leq C(n, \mathcal{C})$$

where  $A_t^\#$  is the shape operator of  $\Sigma_t$  at  $X_t$  and  $\nabla_{\Sigma_t} A_t^\#$  is the covariant derivative of  $A_t^\#$  (with respect to  $\Sigma_t$ ). Note that  $F$  is homogeneous of degree 1,  $\frac{\partial F}{\partial S_i^j}$  is of degree 0 and  $\frac{\partial^2 F}{\partial S_i^j \partial S_k^l}$  is of degree  $-1$ .

Now let  $\tilde{\Sigma}^n$  to be any other  $F$  self-shrinker which is also  $C^5$  asymptotic to  $\mathcal{C}$  at infinity. By the same limiting behavior,  $\tilde{\Sigma}$  is  $C^5$  close to  $\Sigma$  (in the blow-down sense), and hence it can be regarded as a normal graph of a function  $h$  over  $\Sigma$  outside a large ball  $B_R^{n+1}$ . Later we would derive an elliptic equation which is satisfied by  $h$ . To this end, we need the following two lemmas (Lemma 3.1 & Lemma 3.3). The first one gives the decay rate of the function  $h$  and the difference of the shape operators between  $\Sigma$  and  $\tilde{\Sigma}$  as  $|X| \nearrow \infty$ ; in the second lemma, we estimate the coefficients of the differential equation to be satisfied by  $h$ .

**Lemma 3.1.** *There exists  $R = R(\Sigma, \tilde{\Sigma}, n, \mathcal{C}, \|F\|_{C^3(U)}) \geq 1$  so that outside a compact set,  $\tilde{\Sigma}$  is a normal graph over  $\Sigma \setminus \bar{B}_R$  and can be parametrized as*

$$\tilde{X} = X + hN \text{ for } X \in \Sigma \setminus \bar{B}_R$$

where  $N$  is the unit-normal of  $\Sigma$  and  $h$  is the deviation of  $\tilde{\Sigma}$  from  $\Sigma$ . Besides, there hold

$$(3.16) \quad \| |X| h \|_{L^\infty(\Sigma \setminus \bar{B}_R)} + \| |X|^2 \nabla_\Sigma h \|_{L^\infty(\Sigma \setminus \bar{B}_R)} + \| |X|^3 \nabla_\Sigma^2 h \|_{L^\infty(\Sigma \setminus \bar{B}_R)} \leq C(n, \mathcal{C}, \|F\|_{C^3(U)})$$

$$(3.17) \quad \| |X|^3 (\tilde{A}^\# - A^\#) \|_{L^\infty(\Sigma \setminus \bar{B}_R)} + \| |X|^4 (\nabla_\Sigma \tilde{A}^\# - \nabla_\Sigma A^\#) \|_{L^\infty(\Sigma \setminus \bar{B}_R)} \leq C(n, \mathcal{C}, \|F\|_{C^3(U)})$$

$$(3.18) \quad \| |X|^3 \nabla_\Sigma^2 \tilde{A}^\# \|_{L^\infty(\Sigma \setminus \bar{B}_R)} \leq C(n, \mathcal{C}, \|F\|_{C^3(U)})$$

where  $\tilde{A}^\#$  is the shape operator of  $\tilde{\Sigma}$  at  $\tilde{X} = X + hN$  and  $\nabla_\Sigma \tilde{A}^\#$  is the covariant derivative of  $\tilde{A}^\#$  (which can be regarded as a 2-tensor on  $\Sigma$  via the normal graphic parametrization) with respect to  $\Sigma$ .

*Proof.* Choose  $R \gg 1$  (depending on  $\Sigma, \tilde{\Sigma}, n, \mathcal{C}, \|F\|_{C^3(U)}$ ) so that  $\Sigma \setminus \bar{B}_R$  and  $\tilde{\Sigma} \setminus \bar{B}_R$  have the local graph coordinates over tangent hyperplanes of  $\mathcal{C}$  with appropriate estimates for the graphs as before. That is, for each  $\hat{X} \in \Sigma \setminus \bar{B}_R$ , we can respectively parametrize  $\Sigma$  and  $\tilde{\Sigma}$  locally (near  $\Pi(\hat{X}) = \hat{X}_\mathcal{C} \in \mathcal{C}$ ) by

$$X = X(x) \equiv \Pi(\hat{X}) + (x, \mathbf{u}(x))$$

$$\tilde{X} = \tilde{X}(x) \equiv \Pi(\hat{X}) + (x, \tilde{\mathbf{u}}(x))$$

for  $x = (x_1, \dots, x_n) \in B_{\rho|\Pi(\hat{X})|}^n \subset T_{\Pi(\hat{X})}\mathcal{C}$ , which satisfy (by (3.1), (3.2), (3.3) and the comparability of  $|\hat{X}|$  and  $|\hat{X}_C|$ )

$$(3.19) \quad \begin{aligned} & \{ |\hat{X}|^{-1} \| \mathbf{u} \|_{L^\infty(B_{\rho|\Pi(\hat{X})|}^n)} + \| \partial_x \mathbf{u} \|_{L^\infty(B_{\rho|\Pi(\hat{X})|}^n)} + |\hat{X}| \| \partial_x^2 \mathbf{u} \|_{L^\infty(B_{\rho|\Pi(\hat{X})|}^n)} + \dots \\ & + |\hat{X}|^4 \| \partial_x^5 \mathbf{u} \|_{L^\infty(B_{\rho|\Pi(\hat{X})|}^n)} \} \leq C(n, \mathcal{C}) \end{aligned}$$

$$(3.20) \quad \begin{aligned} & \{ |\hat{X}|^{-1} \| \tilde{\mathbf{u}} \|_{L^\infty(B_{\rho|\Pi(\hat{X})|}^n)} + \| \partial_x \tilde{\mathbf{u}} \|_{L^\infty(B_{\rho|\Pi(\hat{X})|}^n)} + |\hat{X}| \| \partial_x^2 \tilde{\mathbf{u}} \|_{L^\infty(B_{\rho|\Pi(\hat{X})|}^n)} + \dots \\ & + |\hat{X}|^4 \| \partial_x^5 \tilde{\mathbf{u}} \|_{L^\infty(B_{\rho|\Pi(\hat{X})|}^n)} \} \leq C(n, \mathcal{C}) \end{aligned}$$

Also, by applying the triangle inequality to (3.10), we get

$$(3.21) \quad \begin{aligned} & \{ |\hat{X}| \| \tilde{\mathbf{u}} - \mathbf{u} \|_{L^\infty(B_{\rho|\Pi(\hat{X})|}^n)} + |\hat{X}|^2 \| \partial_x \tilde{\mathbf{u}} - \partial_x \mathbf{u} \|_{L^\infty(B_{\rho|\Pi(\hat{X})|}^n)} + |\hat{X}|^3 \| \partial_x^2 \tilde{\mathbf{u}} - \partial_x^2 \mathbf{u} \| \\ & + |\hat{X}|^4 \| \partial_x^3 \tilde{\mathbf{u}} - \partial_x^3 \mathbf{u} \|_{L^\infty(B_{\rho|\Pi(\hat{X})|}^n)} \} \leq C(n, \mathcal{C}, \| F \|_{C^3(U)}) \end{aligned}$$

By (3.21), we may assume that  $\tilde{\Sigma}$  is a normal graph of  $h$  defined on  $\Sigma \setminus \bar{B}_R$ ; that is, for each  $x \in B_{\frac{\rho}{2}|\Pi(\hat{X})|}^n \subset T_{\Pi(\hat{X})}\mathcal{C}$ , there is a unique  $y \in B_{\rho|\Pi(\hat{X})|}^n \subset T_{\Pi(\hat{X})}\mathcal{C}$  such that

$$(3.22) \quad \Pi(\hat{X}) + (x, \mathbf{u}(x)) + h(x) \frac{(-\partial_x \mathbf{u}, 1)}{\sqrt{1 + |\partial_x \mathbf{u}|^2}} = \Pi(\hat{X}) + (y, \tilde{\mathbf{u}}(y))$$

or equivalently,

$$\left( x - h(x) \frac{\partial_x \mathbf{u}}{\sqrt{1 + |\partial_x \mathbf{u}|^2}}, \mathbf{u}(x) + \frac{h(x)}{\sqrt{1 + |\partial_x \mathbf{u}|^2}} \right) = (y, \tilde{\mathbf{u}}(y))$$

where  $\frac{(-\partial_x \mathbf{u}, 1)}{\sqrt{1 + |\partial_x \mathbf{u}|^2}}$  is the unit normal  $N$  of  $\Sigma$  at  $X(x) = \Pi(\hat{X}) + (x, \mathbf{u}(x))$ . In other words,  $h$  is defined implicitly by the following equation

$$(3.23) \quad \tilde{\mathbf{u}}(\psi(x)) - \left( \mathbf{u} + \frac{h(x)}{\sqrt{1 + |\partial_x \mathbf{u}|^2}} \right) = 0$$

where

$$(3.24) \quad \psi(x) = x - h(x) \frac{\partial_x \mathbf{u}}{\sqrt{1 + |\partial_x \mathbf{u}|^2}}$$

defines a map from  $B_{\frac{\rho}{2}|\Pi(\hat{X})|}^n \subset T_{\Pi(\hat{X})}\mathcal{C}$  into  $B_{\rho|\Pi(\hat{X})|}^n \subset T_{\Pi(\hat{X})}\mathcal{C}$ . Since  $|h(x)|$  stands for the distance from the point in (3.22):

$$\tilde{X}(\psi(x)) = \Pi(\hat{X}) + (\psi(x), \tilde{\mathbf{u}}(\psi(x)))$$

to  $\Sigma$ , we immediately have

$$|h(x)| \leq |\tilde{\mathbf{u}}(\psi(x)) - \mathbf{u}(\psi(x))| \leq C(n, \mathcal{C}, \| F \|_{C^3(U)}) |\hat{X}|^{-1}$$

To proceed further, notice that for the unit normal vectors of  $\Sigma$  and  $\tilde{\Sigma}$ :

$$(3.25) \quad N(x) = \frac{(-\partial_x \mathbf{u}, 1)}{\sqrt{1 + |\partial_x \mathbf{u}|^2}}, \quad \tilde{N}(x) = \frac{(-\partial_x \tilde{\mathbf{u}}, 1)}{\sqrt{1 + |\partial_x \tilde{\mathbf{u}}|^2}}$$

we may assume (by (3.21), (3.19)) that

$$\| \tilde{N} - N \|_{L^\infty(B_{\rho|\Pi(\hat{X})|}^n)} + \| N \circ \psi - N \|_{L^\infty(B_{\frac{\rho}{2}|\Pi(\hat{X})|}^n)} \leq \frac{1}{3}$$

which implies that for each  $x \in B_{\frac{\rho}{2}|\Pi(\hat{X})|}^n \subset T_{\Pi(\hat{X})}\mathcal{C}$ ,

$$(3.26) \quad \begin{aligned} \tilde{N}(\psi(x)) \cdot N(x) &\geq N(x) \cdot N(x) - |\tilde{N}(\psi(x)) - N(x)| |N(x)| \\ &\geq 1 - \left( |\tilde{N}(\psi(x)) - N(\psi(x))| + |N(\psi(x)) - N(x)| \right) \geq \frac{2}{3} \end{aligned}$$

Let

$$\Theta(x, s) = \tilde{\mathbf{u}} \left( x - s \frac{\partial_x \mathbf{u}}{\sqrt{1 + |\partial_x \mathbf{u}|^2}} \right) - \left( \mathbf{u} + \frac{s}{\sqrt{1 + |\partial_x \mathbf{u}|^2}} \right)$$

then by (3.23), (3.24) and (3.26), we have  $\Theta(x, h(x)) = 0$  and

$$\partial_s \Theta(x, h(x)) = -\sqrt{1 + |\partial_y \tilde{\mathbf{u}}(\psi(x))|^2} \tilde{N}(\psi(x)) \cdot N(x) \leq -\frac{2}{3}$$

Therefore, by the implicit function theorem, we have  $h \in C^2 \left( B_{\frac{\rho}{2}|\Pi(\hat{X})|}^n \right)$ . Besides, by doing the implicit differentiation of (3.23) (or equivalently  $\Theta(x, h(x)) = 0$ ), we get

$$(3.27) \quad \begin{aligned} \frac{1 + \partial_j \tilde{\mathbf{u}} \circ \psi \cdot \partial_j \mathbf{u}}{\sqrt{1 + |\partial_x \mathbf{u}|^2}} \partial_i h &= (\partial_i \tilde{\mathbf{u}} \circ \psi - \partial_i \mathbf{u}) \\ &\quad - \left( \partial_j \tilde{\mathbf{u}} \circ \psi \cdot \partial_i \frac{\partial_j \mathbf{u}}{\sqrt{1 + |\partial_x \mathbf{u}|^2}} + \partial_j \mathbf{u} \frac{\partial_{ij}^2 \mathbf{u}}{(1 + |\partial_x \mathbf{u}|^2)^{\frac{3}{2}}} \right) h \end{aligned}$$

in which we sum over repeated indices. Note that we can use (3.27), together with (3.19) and (3.21), to estimate  $\partial_x h$ . For instance, for the first term on the RHS of the equation, we have

$$\begin{aligned} |\partial_i \tilde{\mathbf{u}} \circ \psi - \partial_i \mathbf{u}| &\leq |\partial_i \tilde{\mathbf{u}} \circ \psi - \partial_i \mathbf{u} \circ \psi| + |\partial_i \mathbf{u} \circ \psi - \partial_i \mathbf{u}| \\ &\leq C(n, \mathcal{C}, \|F\|_{C^3(U)}) |\hat{X}|^{-2} + \sum_j \int_0^1 \left| \partial_{ij}^2 \mathbf{u} \left( x - \theta h \frac{\partial_x \mathbf{u}}{\sqrt{1 + |\partial_x \mathbf{u}|^2}} \right) \right| d\theta \frac{|\partial_j \mathbf{u}|}{\sqrt{1 + |\partial_x \mathbf{u}|^2}} |h| \\ &\leq C(n, \mathcal{C}, \|F\|_{C^3(U)}) |\hat{X}|^{-2} \end{aligned}$$

Thus we get

$$\| \partial_x h \|_{L^\infty(B_{\frac{\rho}{2}|\Pi(\hat{X})|}^n)} \leq C(n, \mathcal{C}, \|F\|_{C^3(U)}) |\hat{X}|^{-2}$$

Similarly, doing the implicit differentiation of (3.27) and using (3.19) and (3.21) yields

$$\| \partial_x^2 h \|_{L^\infty(B_{\frac{\rho}{2}|\Pi(\hat{X})|}^n)} \leq C(n, \mathcal{C}, \|F\|_{C^3(U)}) |\hat{X}|^{-3}$$

The bounds on the covariant derivatives of  $h$  follow from the the following estimates on the pull-back metric  $g_{ij} = \partial_i X \cdot \partial_j X$  and the Christoffel symbols  $\Gamma_{ij}^k$  in (3.11) associated with the local coordinates  $x = (x_1, \dots, x_n)$ :

$$(3.28) \quad \delta_{ij} \leq g_{ij} = 1 + \partial_i \mathbf{u} \partial_j \mathbf{u} \leq \frac{5}{4} \delta_{ij}$$

$$(3.29) \quad |\Gamma_{ij}^k| = \frac{|\partial_k \mathbf{u}|}{1 + |\partial_x \mathbf{u}|^2} |\partial_{ij}^2 \mathbf{u}| \leq C(n, \mathcal{C}, F) |\hat{X}|^{-1}$$

where we have used (3.19). This completes the derivation of (3.16).

As for (3.17), notice that the normal graph reparametrization of  $\tilde{\Sigma}$  amounts to the following change of variables:

$$(3.30) \quad \tilde{X} = \Pi(\hat{X}) + (y, \tilde{\mathbf{u}}(y)) \quad \text{with} \quad y = \psi(x) = x - h(x) \frac{\partial_x \mathbf{u}}{\sqrt{1 + |\partial_x \mathbf{u}|^2}}$$

So from (3.30), (3.19) and (3.16), we have

$$(3.31) \quad \frac{\partial y_k}{\partial x_i} = \delta_i^k - h \cdot \partial_{x_i} \left( \frac{\partial_{x_j} \mathbf{u}}{\sqrt{1 + |\partial_x \mathbf{u}|^2}} \right) - \partial_{x_i} h \frac{\partial_k \mathbf{u}}{\sqrt{1 + |\partial_x \mathbf{u}|^2}} = \delta_i^k + O(|\hat{X}|^{-2})$$

By taking  $R$  sufficiently large, we may assume that  $\psi : B_{\frac{\rho}{2}|\Pi(\hat{X})|}^n \rightarrow \text{Im} \psi \subset B_{\rho|\Pi(\hat{X})|}^n$  is a  $C^2$  diffeomorphism and the inverse of  $\frac{\partial y_k}{\partial x_i}$  satisfies

$$\frac{\partial x_i}{\partial y_k} = \delta_k^i + O(|\hat{X}|^{-2})$$

It follows that the components of shape operators  $\tilde{A}^\#$  of  $\tilde{\Sigma}$  and  $A^\#$  of  $\Sigma$  with respect to the local coordinates  $x = (x_1, \dots, x_n)$  are respectively equal to

$$(3.32) \quad \tilde{A}_i^j = \frac{\partial y_k}{\partial x_i} \frac{\partial x_j}{\partial y_l} \partial_{y_k} \left( \frac{\partial_{y_l} \tilde{\mathbf{u}}}{\sqrt{1 + |\partial_y \tilde{\mathbf{u}}|^2}} \right) \Big|_{y=\varphi(x)}, \quad A_i^j = \partial_{x_i} \left( \frac{\partial_{x_j} \mathbf{u}}{\sqrt{1 + |\partial_x \mathbf{u}|^2}} \right)$$

in which we sum over repeated indices. Using the triangle inequality, combined with (3.19), (3.21), (3.30), (3.16) and (3.31), we then get from (3.32) that

$$|\tilde{A}_i^j - A_i^j| \leq C(n, \mathcal{C}, \|F\|_{C^3(U)}) |\hat{X}|^{-3}$$

Due to (3.28), the above implies that

$$|\tilde{A}^\# - A^\#| \leq C(n, \mathcal{C}, \|F\|_{C^3(U)}) |\hat{X}|^{-3}$$

Also, in view of  $\nabla_\Sigma \tilde{A}^\# \sim \nabla_r \tilde{A}_i^j$ ,  $\nabla_\Sigma A^\# \sim \nabla_r A_i^j$  and

$$(3.33) \quad \nabla_r \tilde{A}_i^j = \partial_r \tilde{A}_i^j - \Gamma_{ri}^s \tilde{A}_s^j + \Gamma_{rs}^j \tilde{A}_i^s, \quad \nabla_r A_i^j = \partial_r A_i^j - \Gamma_{ri}^s A_s^j + \Gamma_{rs}^j A_i^s$$

(in which we sum over repeated indices), we can similarly derive

$$|\nabla_\Sigma \tilde{A}^\# - \nabla_\Sigma A^\#| \leq C(n, \mathcal{C}, \|F\|_{C^3(U)}) |\hat{X}|^{-4}$$

This completes (3.17).

(3.18) follows from taking one more derivative of (3.33) and use (3.32), (3.29), (3.19), (3.21) and (3.28).  $\square$

Next, we would like to define a 2-tensor  $\mathbf{a}$  on  $\Sigma$  (outside a large ball), which would be served as the coefficients of the differential equation to be satisfied by the deviation  $h$ . Note that by (3.12), Lemma 3.1 (in particular (3.17)), we may assume that

$$(3.34) \quad (1 - \theta) |X| A^\# + \theta |X| \tilde{A}^\# \in U \quad \forall X \in \Sigma \setminus \bar{B}_R, \theta \in [0, 1]$$

where  $\tilde{A}^\#$  is the shape operator of  $\tilde{\Sigma}$  at  $\tilde{X} = X + hN$ .

**Definition 3.2.** In the setting of Lemma 3.1, let's take a local coordinate  $x = (x_1, \dots, x_n)$  of  $\Sigma$  (outside a larger ball) so that  $\Sigma$  and  $\tilde{\Sigma}$  can be respectively parametrized as

$$X = X(x), \quad \tilde{X}(x) = X(x) + h(x)N(x)$$

where  $h(x)$  is the deviation and  $N(x)$  is the unit-normal of  $\Sigma$  at  $X(x)$ . Then we define

$$\bar{\mathbf{a}}^{ij}(x) = \sum_k \bar{\mathbf{a}}_k^i(x) g^{kj}(x)$$

$$\bar{\mathbf{a}}_j^i(x) = \int_0^1 \frac{\partial F}{\partial S_i^j} \left( (1-\theta)|X|A^\#(x) + \theta|X|\tilde{A}^\#(x) \right) d\theta$$

and its symmetrization

$$\mathbf{a}^{ij}(x) = \frac{1}{2} (\bar{\mathbf{a}}^{ij}(x) + \bar{\mathbf{a}}^{ji}(x))$$

where  $g^{ij}(x)$  is the inverse of the pull-back metric  $g_{ij} = \partial_i X \cdot \partial_j X$ ,  $A^\#(x) \sim A_i^j(x) = -\partial_i N \cdot \partial_j X$  is the shape operator of  $\Sigma$  at  $X(x)$ ,  $\tilde{A}_t^\#(x) \sim \tilde{A}_i^j(x, t) = -\partial_i \tilde{N} \cdot \partial_j \tilde{X}$  is the shape operator of  $\tilde{\Sigma}$  at  $\tilde{X}(x)$  with  $\tilde{N}(x)$  being the unit-normal of  $\tilde{\Sigma}$  at  $\tilde{X}(x)$ .

Note that

$$\bar{\mathbf{a}}_j^i(x) = \int_0^1 \frac{\partial F}{\partial S_i^j} \left( (1-\theta)|X|A^\#(x) + \theta|X|\tilde{A}^\#(x) \right) d\theta$$

$$= \int_0^1 \frac{\partial F}{\partial S_i^j} \left( (1-\theta)A^\#(x) + \theta\tilde{A}^\#(x) \right) d\theta$$

since  $\frac{\partial F}{\partial S_i^j}$  is homogeneous of degree 0; besides, the operator  $\mathbf{a}$  is independent of the choice of local coordinates and hence defines a 2-tensor on  $\Sigma$ .

We have the following estimates for the tensor  $\mathbf{a}$ , which is based on (3.13), (3.14), (3.15), (3.17), (3.18) and the homogeneity of  $F$  and its derivatives.

**Lemma 3.3.** *There exists  $R = R(\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \|F\|_{C^3(U)}, \lambda, \varkappa) \geq 1$  such that*

$$(3.35) \quad \frac{\lambda}{3} \leq \mathbf{a} \leq \frac{3}{\lambda}$$

$$(3.36) \quad |X| |\nabla_\Sigma \mathbf{a}| \leq 3\varkappa$$

$$(3.37) \quad |X|^2 |\nabla_\Sigma^2 \mathbf{a}| \leq C(n, \mathcal{C}, \|F\|_{C^3(U)})$$

for all  $X \in \Sigma \setminus \bar{B}_R$ .

*Proof.* By (3.13), (3.14), (3.34), (3.17), the homogeneity and continuity of  $F$  (and its derivatives), there exists  $R = R(\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \|F\|_{C^3(U)}, \lambda, \varkappa) \geq 1$  such that

$$\frac{\lambda}{3} \delta_j^i \leq \bar{\mathbf{a}}_j^i = \int_0^1 \frac{\partial F}{\partial S_i^j} \left( (1-\theta)|X|A^\# + \theta|X|\tilde{A}^\# \right) d\theta \leq \frac{3}{\lambda} \delta_j^i$$

$$|X| |\nabla_r \bar{\mathbf{a}}_i^j| = |X| \left| \int_0^1 \sum_{k,l} \frac{\partial^2 F}{\partial S_i^j \partial S_k^l} \left( (1-\theta)A^\# + \theta\tilde{A}^\# \right) \cdot \left( (1-\theta)\nabla_r A_k^l + \theta\nabla_r \tilde{A}_k^l \right) d\theta \right|$$

$$= \left| \int_0^1 \sum_{k,l} \frac{\partial^2 F}{\partial S_i^j \partial S_k^l} \left( (1-\theta) |X| A^\# + \theta |X| \tilde{A}^\# \right) \cdot \left( (1-\theta) |X|^2 \nabla_r A_k^l + \theta |X|^2 \nabla_r \tilde{A}_k^l \right) d\theta \right| \leq 3\kappa$$

Likewise, with the help of (3.15), (3.18), we get

$$|X|^2 \left| \nabla_\Sigma^2 \bar{\mathbf{a}} \right| \leq C(n, \mathcal{C}, \|F\|_{C^3(U)})$$

The conclusion follows immediately.  $\square$

Now we are in a position to derive an equation for  $h$ .

**Proposition 3.4.** *There exists  $R = R(\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \|F\|_{C^3(U)}, \lambda, \kappa) \geq 1$  such that the deviation  $h$  satisfies*

$$(3.38) \quad \nabla_\Sigma \cdot (\mathbf{a} dh) - \frac{1}{2} (X \cdot \nabla_\Sigma h - h) = O(|X|^{-1}) |\nabla_\Sigma h| + O(|X|^{-2}) |h|$$

for  $X \in \Sigma \setminus \bar{B}_R$ , where

$$\nabla_\Sigma \cdot (\mathbf{a} dh) = \sum_{i,j} \nabla_i (\mathbf{a}^{ij} \nabla_j h)$$

and the notation  $O(|X|^{-1})$  means

$$\left| O(|X|^{-1}) \right| \leq C(n, \mathcal{C}, \|F\|_{C^3(U)}) |X|^{-1}$$

*Proof.* Fix  $\hat{X} \in \Sigma \setminus \bar{B}_R$  and take a local coordinate  $x = (x_1, \dots, x_n)$  of  $\Sigma$  which is normal and principal (w.r.t.  $\Sigma$ ) at  $\hat{X} = X(0)$ . That is

$$g_{ij} \Big|_{x=0} = \delta_{ij}, \quad \Gamma_{ij}^k \Big|_{x=0} = 0, \quad A_i^j \Big|_{x=0} = \kappa_i \delta_{ij}$$

where  $g_{ij}$  is the pull-back metric,  $\Gamma_{ij}^k$  is the Christoffel symbols and  $A_i^j$  is the shape operator of  $\Sigma$  at  $X(x)$ . Denote the principal direction of  $\Sigma$  at  $\hat{X}$  by

$$\partial_i X \Big|_{x=0} = e_i$$

Throughout the proof, we adopt the Einstein summation convention (i.e. summing over repeated indices). Recall that we regard  $\tilde{\Sigma}$  (outside a large ball) as a normal graph over  $\Sigma \setminus \bar{B}_R$  and parametrize it by  $\tilde{X} = X(x) + h(x)N(x)$ . We then want to compute some geometric quantities of  $\tilde{\Sigma}$  in terms of this local coordinate at  $\tilde{X}(0) = \hat{X} + hN \Big|_{\hat{X}}$ . First, let's compute

$$(3.39) \quad \begin{aligned} \partial_i \tilde{X} \Big|_{x=0} &= (\delta_i^k - A_i^k h) \partial_k X + \partial_i h N \Big|_{x=0} = (1 - \kappa_i h) e_i + \nabla_i h N \\ \partial_{ij}^2 \tilde{X} \Big|_{x=0} &= -(A_i^k \nabla_j h + A_j^k \nabla_i h + \nabla_i A_j^k \cdot h) e_k + (A_{ij} + \nabla_{ij}^2 h - A_{ij}^2 h) N \end{aligned}$$

which (together with Lemma 3.1) gives the metric of  $\tilde{\Sigma}$ , its inverse and determinant as follows:

$$(3.40) \quad \begin{aligned} \tilde{g}_{ij} \Big|_{x=0} &= (1 - \kappa_i h)^2 \delta_{ij} + \nabla_i h \nabla_j h = (1 - \kappa_i h)^2 \left( \delta_{ij} + \frac{\nabla_i h \nabla_j h}{(1 - \kappa_i h)^2} \right) \\ \tilde{g}^{ij} \Big|_{x=0} &= (1 - \kappa_i h)^{-2} \left( \delta_{ij} + \frac{\nabla_i h \nabla_j h}{(1 - \kappa_i h)^2} \right)^{-1} \end{aligned}$$

$$\begin{aligned}
&= (1 + 2\kappa_i h) \delta^{ij} + O(|\hat{X}|^{-2}) |\nabla_\Sigma h| + O(|\hat{X}|^{-3}) |h| \\
\det \tilde{g} \Big|_{x=0} &= (1 - \kappa_1 h)^2 \cdots (1 - \kappa_n h)^2 \det \left( \delta_{ij} + \frac{\nabla_i h \nabla_j h}{(1 - \kappa_i h)^2} \right) \\
&= 1 - 2Hh + O(|\hat{X}|^{-2}) |\nabla_\Sigma h| + O(|\hat{X}|^{-3}) |h|
\end{aligned}$$

and also the unit-normal of  $\tilde{\Sigma}$ :

$$\begin{aligned}
(3.41) \quad \tilde{N} \Big|_{x=0} &= (\det \tilde{g})^{-\frac{1}{2}} \partial_1 \tilde{X} \wedge \cdots \wedge \partial_n \tilde{X} \\
&= (\det \tilde{g})^{-\frac{1}{2}} \left( - \sum_{i=1}^n \left( \nabla_i h \prod_{j \neq i} (1 - \kappa_j h) \right) e_i + (1 - \kappa_1 h) \cdots (1 - \kappa_n h) N \right) \\
&= - \sum_{i=1}^n \left( 1 + \kappa_i h + O(|\hat{X}|^{-2}) |\nabla_\Sigma h| + O(|\hat{X}|^{-3}) |h| \right) \nabla_i h \cdot e_i \\
&\quad + \left( 1 + O(|\hat{X}|^{-2}) |\nabla_\Sigma h| + O(|\hat{X}|^{-3}) |h| \right) N
\end{aligned}$$

By (3.39), (3.40), (3.41) and Lemma 3.1, we compute the shape operator of  $\tilde{\Sigma}$  at  $\tilde{X}(0)$ :

$$\begin{aligned}
(3.42) \quad \tilde{A}_i^j \Big|_{x=0} &= \tilde{A}_{ik} \tilde{g}^{kj} = \left( \partial_{ik}^2 \tilde{X} \cdot \tilde{N} \right) \tilde{g}^{kj} \\
&= \left( A_{ik} + \nabla_{ik}^2 h + O(|\hat{X}|^{-2}) |\nabla_\Sigma h| + O(|\hat{X}|^{-2}) |h| \right) \left( (1 + 2\kappa_j h) \delta^{kj} + O(|\hat{X}|^{-2}) |\nabla_\Sigma h| \right) \\
&\quad + \left( A_{ik} + \nabla_{ik}^2 h + O(|\hat{X}|^{-2}) |\nabla_\Sigma h| + O(|\hat{X}|^{-2}) |h| \right) O(|\hat{X}|^{-3}) |h| \\
&= A_i^j + \delta^{kj} \nabla_{ik}^2 h + O(|\hat{X}|^{-2}) (|\nabla_\Sigma h| + |h|)
\end{aligned}$$

and

$$(3.43) \quad \tilde{X} \cdot \tilde{N} \Big|_{x=0} = X \cdot N - X \cdot \nabla_\Sigma h + h + O(|\hat{X}|^{-1}) |\nabla_\Sigma h| + O(|\hat{X}|^{-2}) |h|$$

Thus, in view of the  $F$  self-shrinker equation satisfied by  $\Sigma$  and  $\tilde{\Sigma}$ , we get

$$\begin{aligned}
(3.44) \quad 0 &= F(\tilde{A}^\#) - F(A^\#) + \frac{1}{2} (\tilde{X} \cdot \tilde{N} - X \cdot N) \Big|_{x=0} \\
&= \int_0^1 \frac{\partial F}{\partial S_i^j} \left( (1 - \theta) A^\# + \theta \tilde{A}^\# \right) d\theta \cdot (\tilde{A}_i^j - A_i^j) - \frac{1}{2} (X \cdot \nabla_\Sigma h - h) \\
&\quad + O(|\hat{X}|^{-1}) |\nabla_\Sigma h| + O(|\hat{X}|^{-2}) |h| \\
&= \bar{a}_j^i \delta^{jk} \nabla_{ik}^2 h - \frac{1}{2} (X \cdot \nabla_\Sigma h - h) + O(|\hat{X}|^{-1}) |\nabla_\Sigma h| + O(|\hat{X}|^{-2}) |h| \\
&= \bar{a}^{ik} \nabla_{ik}^2 h - \frac{1}{2} (X \cdot \nabla_\Sigma h - h) + O(|\hat{X}|^{-1}) |\nabla_\Sigma h| + O(|\hat{X}|^{-2}) |h| \\
&= \langle \bar{a}, \nabla_\Sigma^2 h \rangle - \frac{1}{2} (X \cdot \nabla_\Sigma h - h) + O(|\hat{X}|^{-1}) |\nabla_\Sigma h| + O(|\hat{X}|^{-2}) |h|
\end{aligned}$$

Note that by the symmetry of the Hessian and Lemma 3.3, we have

$$(3.45) \quad \langle \bar{a}, \nabla_\Sigma^2 h \rangle = \bar{a}^{ij} \nabla_{ij}^2 h = \frac{1}{2} (\bar{a}^{ij} + \bar{a}^{ji}) \nabla_{ij}^2 h = \langle \bar{a}, \nabla_\Sigma^2 h \rangle$$

$$= \nabla_i (\mathbf{a}^{ij} \nabla_j h) - (\nabla_i \mathbf{a}^{ij}) \nabla_j h = \nabla_\Sigma \cdot (\mathbf{a} dh) + O(|\hat{X}|^{-1}) |\nabla_\Sigma h|$$

(3.38) follows from combining (3.44) and (3.45).  $\square$

Our goal is to show that  $h$  vanishes on  $\Sigma \setminus \bar{B}_R$  for some  $R \gg 1$ , which would be done in the next section through Carleman's inequality. For that purpose, we first observe that for each  $t \in [-1, 0)$ ,  $\tilde{\Sigma}_t = \sqrt{-t} \tilde{\Sigma}$  is also a normal graph over  $\Sigma_t \setminus \bar{B}_R$  and it can be parametrized as  $\tilde{X}_t = X_t + h_t N_t$ . For the rest of this section, we would show that each  $h_t = h(\cdot, t)$  satisfies a similar equation as that of  $h(\cdot, -1)$  in Proposition 3.4. Due to the property that  $\{\Sigma_t\}_{-1 \leq t < 0}$  forms a  $F$  curvature flow, it turns out that the evolution of  $h_t$  satisfies a parabolic equation. We then give some estimates for the coefficients of the parabolic equations (as in Lemma 3.3), which is crucial for deriving the Carleman's inequality in the next section.

Now fix  $t \in [-1, 0)$  and define a 2-tensor  $\mathbf{a}_t$  on  $\Sigma_t = \sqrt{-t} \Sigma$  as in Definition 3.2. First, take a local coordinate  $x = (x_1, \dots, x_n)$  of  $\Sigma_t$  (outside a large ball) so that  $\Sigma_t$  and  $\tilde{\Sigma}_t$  can be respectively parametrized as

$$X_t = X_t(x), \quad \tilde{X}_t(x) = X_t(x) + h_t(x) N_t(x)$$

We then define

$$\bar{\mathbf{a}}_t^{ij}(x) = \sum_k \bar{\mathbf{a}}_k^i(x, t) g_t^{kj}(x)$$

$$\bar{\mathbf{a}}_j^i(x, t) = \int_0^1 \frac{\partial F}{\partial S_i^j} \left( (1-\theta) A_t^\#(x) + \theta \tilde{A}_t^\#(x) \right) d\theta$$

and its symmetrization

$$\mathbf{a}_t^{ij}(x) = \frac{1}{2} \left( \bar{\mathbf{a}}_t^{ij}(x) + \bar{\mathbf{a}}_t^{ji}(x) \right)$$

where  $g_t^{ij}(x)$  is the inverse of the pull-back metric  $g_{ij}(x, t) = \partial_i X_t(x) \cdot \partial_j X_t(x)$ ,  $A_t^\#(x) \sim A_i^j(x, t) = -\partial_i N_t(x) \cdot \partial_j X_t(x)$  is the shape operator of  $\Sigma_t$  at  $X_t(x)$  with  $N_t(x)$  being the unit-normal of  $\Sigma_t$  at  $X_t(x)$ ,  $\tilde{A}_t^\# \sim \tilde{A}_i^j(x, t) = -\partial_i \tilde{N}_t(x) \cdot \partial_j \tilde{X}_t(x)$  is the shape operator of  $\tilde{\Sigma}_t$  at  $\tilde{X}_t(x)$  with  $\tilde{N}_t(x)$  being the unit-normal of  $\tilde{\Sigma}_t$  at  $\tilde{X}_t(x)$ .

Then we have the following lemma, which is analogous to Proposition 3.4:

**Lemma 3.5.** *There exists  $R = R(\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \|F\|_{C^3(U)}, \lambda, \varkappa) \geq 1$  such that for each  $t \in [-1, 0)$ , the deviation  $h_t$  satisfies*

$$(3.46) \quad \nabla_{\Sigma_t} \cdot (\mathbf{a}_t dh_t) - \frac{1}{2(-t)} (X_t \cdot \nabla_{\Sigma_t} h_t - h_t) = O(|X_t|^{-1}) |\nabla_{\Sigma_t} h_t| + O(|X_t|^{-2}) |h_t|$$

for  $X_t \in \Sigma_t \setminus \bar{B}_R$ , where

$$\nabla_{\Sigma_t} \cdot (\mathbf{a}_t dh_t) = \sum_{i,j} \nabla_i (\mathbf{a}_t^{ij} \nabla_j h_t)$$

and

$$\left| O(|X_t|^{-1}) \right| \leq C(n, \mathcal{C}, \|F\|_{C^3(U)}) |X_t|^{-1}$$

Also, we have

$$\| |X_t| h_t \|_{L^\infty(\Sigma_t \setminus \bar{B}_R)} + \| |X_t|^2 \nabla_{\Sigma_t} h_t \|_{L^\infty(\Sigma_t \setminus \bar{B}_R)} + \| |X_t|^3 \nabla_{\Sigma_t}^2 h_t \|_{L^\infty(\Sigma_t \setminus \bar{B}_R)}$$



$$(3.47) \quad \leq C(n, \mathcal{C}, \|F\|_{C^3(U)})(-t)$$

*Proof.* Fix  $t \in [-1, 0)$  and  $\hat{X}_t \in \Sigma_t \setminus \bar{B}_R$ , then we have  $\hat{X} = \frac{\hat{X}_t}{\sqrt{-t}} \in \Sigma \setminus \bar{B}_R$  and

$$\begin{aligned} \left( \nabla_{\Sigma_t} \cdot (\mathbf{a}_t dh_t) - \frac{1}{2(-t)} (X_t \cdot \nabla_{\Sigma_t} h_t - h_t) \right) \Big|_{\hat{X}_t} &= \frac{1}{\sqrt{-t}} \left( \nabla_{\Sigma} \cdot (\mathbf{a} dh) - \frac{1}{2} (X \cdot \nabla_{\Sigma} h - h) \right) \Big|_{\hat{X}} \\ &= \frac{1}{\sqrt{-t}} \left( O(|\hat{X}|^{-1}) |\nabla_{\Sigma} h| + O(|\hat{X}|^{-2}) |h| \right) \Big|_{\hat{X}_t} \\ &= \left( O(|\hat{X}_t|^{-1}) |\nabla_{\Sigma_t} h_t| + O(|\hat{X}_t|^{-2}) |h_t| \right) \Big|_{\hat{X}_t} \end{aligned}$$

Similarly, to derive (3.47), it suffices to rescale (3.16) to get

$$\begin{aligned} &|\hat{X}_t| |h_t| + |\hat{X}_t|^2 |\nabla_{\Sigma_t} h_t| + |\hat{X}_t|^3 |\nabla_{\Sigma_t}^2 h_t| \Big|_{\hat{X}_t} \\ &= (-t) \left( |\hat{X}| |h| + |\hat{X}|^2 |\nabla_{\Sigma} h| + |\hat{X}|^3 |\nabla_{\Sigma}^2 h| \right) \Big|_{\hat{X}_t} \\ &\leq C(n, \mathcal{C}, \|F\|_{C^3(U)})(-t) \end{aligned}$$

□

Next, we define “normal parametrizations” of the flow:

**Definition 3.6.**  $X_t = X(\cdot, t)$  is called a “normal parametrization” for the motion of a hypersurface  $\{\Sigma_t\}$  provided that

$$\partial_t X_t = F(A^\#)N$$

That is, each particle on the hypersurface moves in normal direction during the flow (see also Definition 2.4).

In the derivation of the parabolic equation to be satisfied by  $h_t = h(\cdot, t)$ , we start with a “radial parametrization” of the flow  $\{\Sigma_t\}_{-1 \leq t < 0}$  (i.e. each particles on the hypersurface moves in the radial direction along the flow, see the proof of Proposition 3.7 for details), then we make a transition to the “normal parametrization” by using a time-dependent tangential diffeomorphism. Note that in general, the “radial parametrization” exists only for a short period of time (unlike the “vertical parametrization”), so later in the proof, we would do a local argument, which is sufficient for deriving the equation.

**Proposition 3.7.** *There exists  $R = R(\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \|F\|_{C^3(U)}, \lambda, \varkappa) \geq 1$  so that in the normal parametrization of the  $F$  curvature flow  $\{\Sigma_t\}_{-1 \leq t < 0}$ , the deviation  $h_t$  satisfies*

$$(3.48) \quad \mathbf{P}h \equiv \partial_t h - \nabla_{\Sigma_t} \cdot (\mathbf{a}(\cdot, t) dh) = O(|X_t|^{-1}) |\nabla_{\Sigma_t} h| + O(|X_t|^{-2}) |h|$$

$$(3.49) \quad h(\cdot, 0) = 0 \quad \text{as } t \nearrow 0$$

for  $X_t \in \Sigma_t \setminus \bar{B}_R$ ,  $-1 \leq t < 0$ , where  $\mathbf{a}(\cdot, t) = \mathbf{a}_t$ .

*Proof.* Fix  $\hat{t} \in [-1, 0)$ ,  $\hat{X} \in \Sigma_{\hat{t}} \setminus \bar{B}_R$ , and take a local coordinate  $x = (x_1, \dots, x_n)$  of  $\Sigma_{\hat{t}}$  around  $\hat{X}$ . Define the “radial parametrization” of the flow starting at time  $\hat{t}$  near the point  $\hat{X}$  by

$$X(x, t) = \frac{\sqrt{-t}}{\sqrt{-\hat{t}}} X_{\hat{t}}(x)$$

For this parametrization, we can decompose the velocity vector into the normal part and the tangential part as follows:

$$\begin{aligned}
 (3.50) \quad \partial_t X(x, t) &= \frac{-1}{2\sqrt{-t}\sqrt{-t}} X_{\hat{t}}(x) \\
 &= \frac{-1}{2\sqrt{-t}\sqrt{-t}} \left( (X_{\hat{t}}(x) \cdot N_{\hat{t}}(x)) N_{\hat{t}}(x) + \sum_{i,j} g_{\hat{t}}^{ij}(x) (X_{\hat{t}}(x) \cdot \partial_j X_{\hat{t}}(x)) \partial_i X_{\hat{t}}(x) \right) \\
 &= F \left( A_i^j(x, t) \right) N(x, t) - \sum_{i,j} \frac{1}{2(-t)} g^{ij}(x, t) (X(x, t) \cdot \partial_j X(x, t)) \partial_i X(x, t)
 \end{aligned}$$

in which we use the  $F$  self-shrinker equation of  $\Sigma_{\hat{t}} = \sqrt{-\hat{t}} \Sigma$  (in Definition 2.4) and the homogeneity of  $F$ . Now consider the following ODE system:

$$(3.51) \quad \partial_t x_i = \sum_{i,j} \frac{1}{2(-t)} g^{ij}(x, t) (X(x, t) \cdot \partial_j X(x, t))$$

$$x_i \Big|_{t=\hat{t}} = \xi_i, \quad i = 1, \dots, n$$

Let the solution (which exists at least for a while) to be  $x = \varphi_t(\xi)$ . In other words,  $\varphi_t$  is the local diffeomorphism on  $\Sigma_t$  generated by the tangent vector field  $\frac{1}{2(-t)} X(x, t)^\top$ . By (3.50) and (3.51), the reparametrization  $X(\varphi_t(\xi), t)$  of the flow becomes a normal parametrization.

On the other hand, in the radial parametrization, we have

$$h(x, t) = \frac{\sqrt{-t}}{\sqrt{-\hat{t}}} h_{\hat{t}}(x)$$

Thus, by (3.51) and Lemma 3.5, we get

$$\begin{aligned}
 \frac{\partial}{\partial t} \{h(\varphi_t(\xi), t)\} &= \partial_t h(x, t) + \sum_{i,j} \frac{1}{2(-t)} g^{ij}(x, t) (X(x, t) \cdot \partial_j X(x, t)) \partial_i h(x, t) \Big|_{x=\varphi_t(\xi)} \\
 &= \frac{1}{2(-t)} \{-h(x, t) + X(x, t) \cdot \nabla_{\Sigma_t} h\} \Big|_{x=\varphi_t(\xi)} \\
 &= \nabla_{\Sigma_t} \cdot (\mathbf{a}(\cdot, t) dh_t) + O(|X_t|^{-1}) |\nabla_{\Sigma_t} h_t| + O(|X_t|^{-2}) |h_t| \Big|_{x=\varphi_t(\xi)}
 \end{aligned}$$

which proves (3.48).

(3.49) follows from (3.47).  $\square$

Lastly, we conclude this section by giving some estimates on the 2-tensor  $\mathbf{a}(\cdot, t)$  for each time-slice  $\Sigma_t$ .

**Proposition 3.8.** *There exists  $R = R(\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \|F\|_{C^3(U)}, \lambda, \varkappa) \geq 1$  so that for  $t \in [-1, 0)$ ,  $X_t \in \Sigma_t \setminus \bar{B}_R$ , there hold*

$$(3.52) \quad \frac{\lambda}{3} \leq \mathbf{a}(\cdot, t) \leq \frac{3}{\lambda}$$

$$(3.53) \quad |X_t| \left| \nabla_{\Sigma_t} \mathbf{a}(\cdot, t) \right| \leq 3\varkappa$$

$$(3.54) \quad |X_t|^2 \left| \nabla_{\Sigma_t}^2 \mathbf{a}(\cdot, t) \right| \leq C(n, \mathcal{C}, \|F\|_{C^3(U)})$$

$$(3.55) \quad |X_t|^2 \left| \partial_t \mathbf{a}(\cdot, t) \right| \leq C(n, \mathcal{C}, \|F\|_{C^3(U)})$$

where the time derivative in the last term is taken with respect to a normal parametrization of the flow  $\{\Sigma_t\}_{-1 \leq t < 0}$ .

*Proof.* We adopt the Einstein summation convention throughout the proof.

By using the rescaling argument and the homogeneity of the derivatives of  $F$ , (3.52), (3.53), (3.54) follow from (3.35), (3.36), (3.37), respectively. As for (3.55), note that in a normal parametrization, we have

$$(3.56) \quad \partial_t \bar{a}^{ij}(t) = \partial_t \left( \bar{a}_k^i(t) g_t^{kj} \right) = (\partial_t \bar{a}_k^i(t)) g_t^{kj} + 2 \bar{a}_k^i(t) F \left( A_t^\# \right) A_t^{kj}$$

in which we use the following evolution equation for the metric along the  $F$  curvature flow  $\{\Sigma_t\}_{-1 \leq t < 0}$  (see [A]):

$$(3.57) \quad \partial_t g_{ij}(t) = -2F \left( A_t^\# \right) A_{ij}(t), \quad \partial_t g_t^{ij} = 2F \left( A_t^\# \right) A_t^{ij}$$

By the rescaling argument, (3.12), and the homogeneity of  $F$  and its derivatives, we can estimate each term in (3.56) by

$$|X_t|^2 \left| F \left( A_t^\# \right) A_t^{ij} \right| = \left| F \left( |X_t| A_t^\# \right) \cdot |X_t| A_t^{ij} \right| \leq C(n, \mathcal{C}, \|F\|_{C^3(U)})$$

and

$$\begin{aligned} |X_t|^2 |\partial_t \bar{a}_j^i| &= |X_t|^2 \left| \int_0^1 \frac{\partial^2 F}{\partial S_i^j \partial S_k^l} \left( (1-\theta) A_t^\# + \theta \tilde{A}_t^\# \right) \cdot \left( (1-\theta) \partial_t A_k^l + \theta \partial_t \tilde{A}_k^l \right) d\theta \right| \\ &= \left| \int_0^1 \frac{\partial^2 F}{\partial S_i^j \partial S_k^l} \left( (1-\theta) |X_t| A_t^\# + \theta |X_t| \tilde{A}_t^\# \right) \cdot \left( (1-\theta) |X_t|^3 \partial_t A_k^l + \theta |X_t|^3 \partial_t \tilde{A}_k^l \right) d\theta \right| \\ &\leq C(n, \mathcal{C}, \|F\|_{C^3(U)}) \left| \int_0^1 \left( (1-\theta) |X_t|^3 \partial_t A_k^l + \theta |X_t|^3 \partial_t \tilde{A}_k^l \right) d\theta \right| \end{aligned}$$

Thus, to establish (3.55), it suffices to show that

$$(3.58) \quad |X_t|^3 |\partial_t A_t^\#| \leq C(n, \mathcal{C}, \|F\|_{C^3(U)})$$

$$(3.59) \quad |X_t|^3 |\partial_t \tilde{A}_t^\# - \partial_t A_t^\#| \leq C(n, \mathcal{C}, \|F\|_{C^3(U)})$$

for all  $X_t \in \Sigma_t \setminus \bar{B}_R$ ,  $t \in [-1, 0)$ .

Firstly, let's recall the evolution equation for the shape operator  $A_t^\#$  in the normal parametrization along the flow (see [A]):

$$(3.60) \quad \begin{aligned} \partial_t A_i^j(t) &= \frac{\partial F}{\partial S_k^l} \left( A_t^\# \right) \cdot g_t^{lm} \nabla_{km}^2 A_i^j + \frac{\partial F}{\partial S_k^l} \left( A_t^\# \right) \cdot (A_t^2)_k^l A_i^j(t) \\ &\quad + \frac{\partial^2 F}{\partial S_k^l \partial S_p^q} \left( A_t^\# \right) \cdot g_t^{jm} \nabla_i A_k^l(t) \nabla_m A_p^q(t) \end{aligned}$$

which yields (3.58) by the rescaling argument, (3.15) and the homogeneity of  $F$  and its derivatives.

Secondly, we would compute  $\partial_t (\tilde{A}_t^\# - A_t^\#)$  in the normal parametrization (of  $\{\Sigma_t\}_{-1 \leq t < 0}$ ) by using the same trick as in the proof of Proposition 3.7. Fix  $\hat{t} \in [-1, 0)$ ,  $\hat{X} \in \Sigma_{\hat{t}} \setminus \bar{B}_R$ , and take a local coordinate  $x = (x_1, \dots, x_n)$  of  $\Sigma_{\hat{t}}$  which is

normal at  $\hat{X} = X(0)$ . Consider the radial parametrization of the flow starting at time  $\hat{t}$  near the point  $\hat{X}$  by

$$X(x, t) = \frac{\sqrt{-t}}{\sqrt{-\hat{t}}} X_{\hat{t}}(x)$$

Then we have

$$\tilde{A}_i^j(x, t) - A_i^j(x, t) = \frac{\sqrt{-\hat{t}}}{\sqrt{-t}} \left( \tilde{A}_i^j(x, \hat{t}) - A_i^j(x, \hat{t}) \right)$$

Let  $x = \varphi_t(\xi)$  with  $\varphi_{\hat{t}} = \text{id}$  to be the local diffeomorphism on  $\Sigma_{\hat{t}}$  generated by the tangent vector field  $\frac{1}{2(-\hat{t})} X(\cdot, \hat{t})^\top$  as before. Then the reparametrization  $X(\varphi_t(\xi), t)$  of the flow becomes a normal parametrization and we have

$$\begin{aligned} (3.61) \quad & \partial_t \left( \tilde{A}_i^j(\varphi_t(\xi), t) - A_i^j(\varphi_t(\xi), t) \right) \Big|_{\xi=0, t=\hat{t}} = \left( \partial_t \tilde{A}_i^j - \partial_t A_i^j \right) (\varphi_t(\xi), t) \\ & + \frac{1}{2(-\hat{t})} g^{kl}(\varphi_t(\xi), t) (X_t(\varphi_t(\xi), t) \cdot \partial_l X_t(\varphi_t(\xi), t)) \left( \partial_k \tilde{A}_i^j(\varphi_t(\xi), t) - \partial_k A_i^j(\varphi_t(\xi), t) \right) \Big|_{\xi=0, t=\hat{t}} \\ & = \frac{1}{2(-\hat{t})} \left\{ \left( \tilde{A}_i^j(\hat{t}) - A_i^j(\hat{t}) \right) + g_{\hat{t}}^{kl} (X_{\hat{t}} \cdot \partial_l X_{\hat{t}}) \left( \nabla_k \tilde{A}_i^j(\hat{t}) - \nabla_k A_i^j(\hat{t}) \right) \right\} \Big|_{\hat{X}} \end{aligned}$$

Note that for each  $t \in [-1, 0)$ , by the rescaling argument and (3.17), we have

$$\begin{aligned} & \| |X_t|^3 \left( \tilde{A}_t^\# - A_t^\# \right) \|_{L^\infty(\Sigma_t \setminus \bar{B}_R)} + \| |X_t|^4 \left( \nabla_{\Sigma_t} \tilde{A}_t^\# - \nabla_{\Sigma_t} A_t^\# \right) \|_{L^\infty(\Sigma_t \setminus \bar{B}_R)} \\ & \leq \left\{ \| |X|^3 \left( \tilde{A}^\# - A^\# \right) \|_{L^\infty(\Sigma \setminus \bar{B}_R)} + \| |X|^4 \left( \nabla_\Sigma \tilde{A}^\# - \nabla_\Sigma A^\# \right) \|_{L^\infty(\Sigma \setminus \bar{B}_R)} \right\} (-t) \\ & \leq C(n, \mathcal{C}, \|F\|_{C^3(U)}) (-t) \end{aligned}$$

Combining (3.61) and (3.62) to get (3.59).  $\square$

#### 4. CARLEMAN'S INEQUALITIES AND UNIQUENESS OF $F$ SELF-SHRINKERS WITH A TANGENT CONE AT INFINITY

This section is a continuation of the previous section. Here we still assume that  $\Sigma^n$  and  $\tilde{\Sigma}^n$  are properly embedded  $F$  self-shrinkers (in Definition 2.4) which are  $C^5$  asymptotic to the cone  $\mathcal{C}^n$  at infinity, and they induce  $F$  curvature flows  $\{\Sigma_t\}_{-1 \leq t \leq 0}$  and  $\{\tilde{\Sigma}_t\}_{-1 \leq t \leq 0}$ , where  $\Sigma_t = \sqrt{-t} \Sigma$ ,  $\tilde{\Sigma}_t = \sqrt{-t} \tilde{\Sigma}$  for  $t \in [-1, 0)$  and  $\Sigma_0 = \mathcal{C} = \tilde{\Sigma}_0$ . We also consider the deviation  $h_t = h(\cdot, t)$  of  $\tilde{\Sigma}_t$  from  $\Sigma_t$  for  $t \in [-1, 0]$  (we set  $h_0 = 0$ ), which is defined on  $\Sigma_t \setminus \bar{B}_R$  (see Lemma 3.1), where  $R \gg 1$  (depending on  $\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \|F\|_{C^3(U)}, \lambda, \varkappa$ ). For the function  $h$ , recall that we have Proposition 3.7 and Proposition 3.8. The Einstein summation convention is adopted throughout this section (i.e. summing over repeated indices).

At the beginning, we would improve the rate of decay of  $h_t$  as  $t \nearrow 0$  in (3.47) to be exponential. To achieve that, we need Proposition 4.5, which is due to [EF] and [N] for different cases. The proof of Proposition 4.5 would be included here for readers' convenience, and it is based on two crucial lemmas (which we state

without proof). The first one is the mean value inequality for parabolic equations from [LSU].

**Lemma 4.1** (Mean value inequality). *Let  $P = \partial_t - \partial_i (a^{ij}(x, t) \partial_j)$  be a differential operator such that  $a_t^{ij} = a^{ij}(\cdot, t) \in C^1(B_1^n)$  for  $t \in [-1, 0]$ ,  $a^{ij} = a^{ji}$ , and*

$$\lambda \delta^{ij} \leq a^{ij} \leq \frac{1}{\lambda} \delta^{ij}$$

$$|a^{ij}(x, t) - a^{ij}(\tilde{x}, \tilde{t})| \leq L \left( |x - \tilde{x}| + |t - \tilde{t}|^{\frac{1}{2}} \right)$$

for some  $\lambda \in (0, 1]$ ,  $L > 0$ , where  $B_1^n = \{x \in \mathbb{R}^n \mid |x| < 1\}$ .

Suppose that  $u \in C^{2,1}(B_1^n \times [-T, 0])$  satisfies

$$|Pu| \leq L \left( \frac{1}{\sqrt{T}} |\partial_x u| + \frac{1}{T} |u| \right)$$

for some  $T \in (0, 1]$ , then there holds

$$|u(x, t)| + \sqrt{-t} |\partial_x u(x, t)| \leq C(n, \lambda, L) \oint_{Q(x, t; \sqrt{-t})} |u|$$

for  $(x, t) \in Q(0, 0; \frac{\sqrt{T}}{2})$ , where  $Q(x, t; r) = B_r^n(x) \times (t - r^2, t]$  is the parabolic cylinder centered at  $(x, t)$  and  $\oint_{\mathcal{D}}$  means taking the average of a function over the region  $\mathcal{D}$ .

*Remark 4.2.* To prove the above lemma, we may consider the following change of variables:

$$(x, t) = (\sqrt{T} \hat{x}, T \hat{t})$$

In the new variables, the equation in Lemma 4.1 becomes

$$\left| \partial_{\hat{t}} u - \partial_{\hat{x}_i} \left( a^{ij}(\sqrt{T} \hat{x}, T \hat{t}) \partial_{\hat{x}_j} u \right) \right| \leq L (|\partial_{\hat{x}} u| + |u|)$$

for  $\hat{x} \in B_{1/\sqrt{T}}^n$ ,  $\hat{t} \in [-1, 0]$ . Then apply standard estimates from [LSU] to the normalized equation (note that  $T \in (0, 1]$ ).

The second lemma is a local type of Carleman's inequalities from [EFV].

**Lemma 4.3** (Local Carleman's inequality). *Let  $P = \partial_t - \partial_i (a^{ij}(x, t) \partial_j)$  be a differential operator such that  $a_t^{ij} = a^{ij}(\cdot, t) \in C^1(B_1^n)$  for  $t \in [-1, 0]$ ,  $a^{ij} = a^{ji}$ ,  $a^{ij}(0, 0) = \delta^{ij}$  and*

$$\lambda \delta^{ij} \leq a^{ij} \leq \frac{1}{\lambda} \delta^{ij}$$

$$|a^{ij}(x, t) - a^{ij}(\tilde{x}, \tilde{t})| \leq L \left( |x - \tilde{x}| + |t - \tilde{t}|^{\frac{1}{2}} \right)$$

for some  $\lambda \in (0, 1]$ ,  $L > 0$ , where  $B_1^n = \{x \in \mathbb{R}^n \mid |x| < 1\}$ .

Then for any fixed constant  $M \geq 4$ , there exists a non-increasing function  $\varphi : (-\frac{4}{M}, 0) \rightarrow \mathbb{R}_+$  satisfying  $\frac{-t}{\sigma} \leq \varphi(t) \leq -t$  for some constant  $\sigma = \sigma(n, \lambda, L) \geq 1$ , so that for any constant  $\delta \in (0, \frac{1}{M})$  and function  $v \in C_c^{2,1}(B_1^n \times (-\frac{2}{M}, 0])$ , there holds

$$M^2 \int v^2 \varphi_\delta^{-M} \Phi_\delta dx dt + M \int |\partial_x v|^2 \varphi_\delta^{1-M} \Phi_\delta dx dt$$

$$\leq \sigma \int |Pv|^2 \varphi_\delta^{1-M} \Phi_\delta dx dt + (\sigma M)^M \sup_{t < 0} \int (|\partial_x v|^2 + v^2) dx + \sigma M \int v^2 \varphi_\delta^{-M} \Phi_\delta dx \Big|_{t=0}$$

where  $\varphi_\delta(t) = \varphi(t - \delta)$  and  $\Phi_\delta(x, t) = \Phi(x, t - \delta) = \frac{1}{(4\pi(-t + \delta))^{\frac{n}{2}}} \exp\left(-\frac{|x|^2}{4(-t + \delta)}\right)$ .

*Remark 4.4.* Note that the last term on the RHS of the above inequality vanishes provided that  $v|_{t=0} = 0$ .

Now we state the proposition (of showing the exponential decay) and include the proof (which is due to [EF] and [N]).

**Proposition 4.5** (Exponential decay and Unique continuation principle). *Let  $P = \partial_t - \partial_i(a^{ij}(x, t)\partial_j)$  be a differential operator such that  $a_t^{ij} = a^{ij}(\cdot, t) \in C^1(B_1^n)$  for  $t \in [-1, 0]$ ,  $a^{ij} = a^{ji}$ , and*

$$\lambda \delta^{ij} \leq a^{ij} \leq \frac{1}{\lambda} \delta^{ij}$$

$$|a^{ij}(x, t) - a^{ij}(\tilde{x}, \tilde{t})| \leq L \left( |x - \tilde{x}| + |t - \tilde{t}|^{\frac{1}{2}} \right)$$

for some  $\lambda \in (0, 1]$ ,  $L > 0$ , where  $B_1^n = \{x \in \mathbb{R}^n \mid |x| < 1\}$ .

Suppose that  $u \in C^{2,1}(B_1^n \times [-T, 0])$  satisfies

$$(4.1) \quad |Pu| \leq L \left( \frac{1}{\sqrt{T}} |\partial_x u| + \frac{1}{T} |u| \right)$$

for some  $T \in (0, 1]$ , and that either  $u$  vanishes at  $(0, 0)$  to infinite order (see [EF]), i.e.

$$(4.2) \quad \forall k \in \mathbb{N} \quad \exists C_k > 0 \quad \text{s.t.} \quad |u(x, t)| \leq C_k (|x| + \sqrt{-t})^k$$

or  $u$  vanishes identically at  $t = 0$  (see [N]), i.e.

$$(4.3) \quad u|_{t=0} = 0$$

Then there exist  $\Lambda = \Lambda(n, \lambda, L) > 0$ ,  $\alpha = \alpha(n, \lambda, L) \in (0, 1)$  so that

$$(4.4) \quad |u(x, t)| + |\partial_x u(x, t)| \leq \Lambda e^{\frac{1}{\Lambda T}} (\|\partial_x u\|_{L^\infty(B_1 \times [-T, 0])} + \|u\|_{L^\infty(B_1 \times [-T, 0])})$$

for  $x \in B_{1/4}^n$ ,  $t \in [-\alpha T, 0)$ .

*Remark 4.6.* Later we would apply Proposition 4.5 under the condition (4.3) to show the exponential decay of the deviation  $h_t = h(\cdot, t)$  as  $t \nearrow 0$ . On the other hand, Proposition 4.5 implies that under the condition (4.2), the function  $u$  in (4.1) must vanish identically at  $t = 0$ . In particular, in the case when  $u$  is time-independent (so  $u$  would satisfy an elliptic equation, and we could take  $t = 0$  in (4.2)), it implies that  $u$  vanishes identically. Such phenomenon is called the “unique continuation principle” and would be used at the end of this section.

*Proof.* For simplicity, we may assume that  $a^{ij}(0, 0) = \delta^{ij}$ . Otherwise, we could do change of variables like  $\hat{x} = a^{ij}(0, 0)^{-\frac{1}{2}} x$  to achieve that.

In the proof, we mainly focus on dealing with the case of (4.2), since the same argument also applies to the case of (4.3) with only a slight modification, which we would point out on the way of proof.

Fix a constant  $M \in [\frac{4L^2(n+\sigma)}{T}, \infty)$  (to be chosen), where  $\sigma = \sigma(n, \lambda, L) \geq 1$  is the constant that appears in Lemma 4.3. Then for any  $\epsilon \in (0, \min\{\frac{1}{M}, 1\})$ , choose smooth cut-off functions  $\zeta = \zeta(x)$ ,  $\eta_\epsilon = \eta_\epsilon(t)$  and  $\eta = \eta(t)$  such that

$$\chi_{B_{1/2}^n} \leq \zeta \leq \chi_{B_1^n}, \quad \|\zeta\|_{C^2} \leq 4$$

$$\chi_{[\frac{-1}{M}, -\epsilon]} \leq \eta_\epsilon \leq \chi_{[\frac{-2}{M}, 0]}, \quad \chi_{[\frac{-1}{M}, 0]} \leq \eta \leq \chi_{[\frac{-2}{M}, 0]}, \quad \eta_\epsilon \nearrow \eta \quad \text{as } \epsilon \searrow 0$$

$$|\partial_t \eta_\epsilon| \leq 2M \chi_{[\frac{-2}{M}, \frac{-1}{M}]} + \frac{2}{\epsilon} \chi_{[-\epsilon, 0]}$$

where  $\chi_{B_1^n}$  is the characteristic function of  $B_1^n$ .

Let  $v_\epsilon(x, t) = \zeta(x) \eta_\epsilon(t) v(x, t)$  be a localization of  $v$ , which satisfies  $v_\epsilon|_{t=0} = 0$  and converges pointwisely to  $v(x, t) = \zeta(x) \eta(t) v(x, t)$  as  $\epsilon \searrow 0$ . Besides, we have

$$(4.5) \quad |Pv_\epsilon| \leq \{L\zeta\eta_\epsilon \left( \frac{1}{\sqrt{T}} |\partial_x v| + \frac{1}{T} |v| \right) + C(\lambda, L) (|\partial_x v| + |v|) \chi_{B_1 \setminus B_{\frac{1}{2}}}(x) + 2LM |v| \chi_{[\frac{-2}{M}, \frac{-1}{M}]}(t) + \frac{2L}{\epsilon} |v| \chi_{[-\epsilon, 0]}(t)\} \\ \leq L \left( \frac{1}{\sqrt{T}} |\partial_x v_\epsilon| + \frac{1}{T} |v_\epsilon| \right) + C(\lambda, L) M (|\partial_x v| + |v|) \chi_E(x, t) + \frac{2L}{\epsilon} |v| \chi_{[-\epsilon, 0]}(t)$$

where  $E = \{(x, t) \in B_1^n \times [-1, 0] \mid \frac{1}{2} \leq |x| \leq 1 \text{ or } \frac{-2}{M} \leq t \leq \frac{-1}{M}\}$ . Note that in the case of (4.3), it suffices to consider  $v$  (without using the  $\epsilon$  cut-off) in order to make the function vanishing at  $t = 0$ .

Then for each  $\delta \in (0, \frac{1}{M})$ , by Lemma 4.3 (applied to  $v_\epsilon$ ) and (4.5), there holds

$$M^2 \int v_\epsilon^2 \varphi_\delta^{-M} \Phi_\delta dx dt + M \int |\partial_x v_\epsilon|^2 \varphi_\delta^{1-M} \Phi_\delta dx dt \\ \leq \{2\sigma L^2 \int \left( \frac{v_\epsilon^2}{T^2} + \frac{|\partial_x v_\epsilon|^2}{T} \right) \varphi_\delta^{1-M} \Phi_\delta dx dt + 2C(\lambda, L) \sigma M^2 \int_E (|\partial_x v|^2 + v^2) \varphi_\delta^{1-M} \Phi_\delta dx dt \\ + \frac{4\sigma L^2}{\epsilon^2} \int_{-\epsilon}^0 \int_{B_1} v^2 \varphi_\delta^{1-M} \Phi_\delta dx dt + (\sigma M)^M \sup_t \int (|\partial_x v_\epsilon|^2 + v_\epsilon^2) dx\}$$

By our choice of  $M$ , the first term on the RHS of the above inequality can be absorbed by its LHS. Thus, we get

$$(4.6) \quad M^2 \int v_\epsilon^2 \varphi_\delta^{-M} \Phi_\delta dx dt \leq \{C(\lambda, L) \sigma M^2 \int_E (|\partial_x v|^2 + v^2) \varphi_\delta^{1-M} \Phi_\delta dx dt \\ + 4(\sigma M)^M \sup_{-T \leq t \leq 0} \int_{B_1} (|\partial_x v|^2 + v^2) dx + \frac{4\sigma L^2}{\epsilon^2} \int_{-\epsilon}^0 \int_{B_1} v^2 \varphi_\delta^{1-M} \Phi_\delta dx dt\}$$

Now choose an integer  $k \geq M + \frac{n}{2}$ , then by (4.2) the last term on the RHS of (4.6) can be estimated by

$$(4.7) \quad \frac{4\sigma L^2}{\epsilon^2} \int_{-\epsilon}^0 \int_{B_1} v^2 \varphi_\delta^{1-M} \Phi_\delta dx dt \\ \leq \frac{4\sigma L^2}{\epsilon^2} \int_{-\epsilon}^0 \int_{B_1} \frac{C_k (|x| + \sqrt{-t})^{2(M + \frac{n}{2})} \exp\left(\frac{-|x|^2}{4(-t+\delta)}\right)}{\left(\frac{-t+\delta}{\sigma}\right)^{M-1} (4\pi(-t+\delta))^{\frac{n}{2}}} dx dt$$

$$\begin{aligned}
&\leq C(n, C_k, \sigma, M, L) \frac{1}{\epsilon^2} \int_{-\epsilon}^0 \left\{ \int_{B_1} \left( \frac{|x|^2}{-t+\delta} + 1 \right)^{M+\frac{n}{2}} \exp \left( \frac{-|x|^2}{4(-t+\delta)} \right) dx \right\} (-t+\delta) dt \\
&\leq C(n, C_k, \sigma, M, L) \frac{1}{\epsilon^2} \int_{-\epsilon}^0 \left\{ \int_0^\infty (|\xi|^2 + 1)^{M+\frac{n}{2}} \exp \left( \frac{-|\xi|^2}{4} \right) d\xi \right\} (-t+\delta)^{\frac{n}{2}+1} dt \\
&\leq C(n, C_k, \sigma, M, L) \frac{(\epsilon+\delta)^{\frac{n}{2}+2} - \delta^{\frac{n}{2}+2}}{\epsilon^2}
\end{aligned}$$

In view of (4.7), apply the monotone convergence theorem to (4.6) by first letting  $\delta \searrow 0$  and then  $\epsilon \searrow 0$  to arrive at

$$\begin{aligned}
(4.8) \quad &\int_{B_{\frac{1}{2}} \times (\frac{-1}{M}, 0)} v^2 \varphi^{-M} \Phi dx dt \\
&\leq C(\Lambda, L) \sigma \int_E (|\partial_x v|^2 + v^2) \varphi^{1-M} \Phi dx dt + (4\sigma M)^M \sup_{-T \leq t \leq 0} \int_{B_1} (|\partial_x v|^2 + v^2) dx \\
&\leq C(n, \Lambda, L) \left( \sigma \int_E \varphi^{1-M} \Phi dx dt + (\sigma M)^M \right) \left( \|\partial_x v\|_{L^\infty(B_1 \times [-T, 0])}^2 + \|v\|_{L^\infty(B_1 \times [-T, 0])}^2 \right)
\end{aligned}$$

Note that in the case of (4.3), we can get (4.8) directly from taking the limit as  $\delta \searrow 0$  without using (4.7).

Next, we would estimate the first term on the RHS of (4.8). For  $(x, t) \in E$ , either  $\frac{-2}{M} \leq t \leq \frac{-1}{M}$ , in which case we have

$$(4.9) \quad \varphi^{1-M} \Phi(x, t) \leq \left( \frac{-t}{\sigma} \right)^{1-M} \frac{1}{(4\pi(-t))^{\frac{n}{2}}} \leq \frac{(\sigma M)^{M-1+\frac{n}{2}}}{(4\pi\sigma)^{\frac{n}{2}}}$$

or  $\frac{1}{2} \leq |x| \leq 1$  and  $\frac{-2}{M} \leq t < 0$ , in which case we have

$$\begin{aligned}
(4.10) \quad &\varphi^{1-M} \Phi(x, t) \leq \left( \frac{\sigma M}{(-t)M} \right)^{M-1} \frac{M^{\frac{n}{2}}}{(4\pi(-t)M)^{\frac{n}{2}}} \exp \left( \frac{-M}{16(-tM)} \right) \\
&= \frac{(\sigma M)^{M-1} \left( \frac{M}{4\pi} \right)^{\frac{n}{2}}}{(-tM)^{M-1+\frac{n}{2}} \exp \left( \frac{M/16}{-tM} \right)} \leq (\sigma M)^{M-1} \left( \frac{M}{4\pi} \right)^{\frac{n}{2}} \left( \frac{M-1+\frac{n}{2}}{e^{M/16}} \right)^{M-1+\frac{n}{2}} \\
&\leq \left( \frac{16\sigma}{e} \left( M-1+\frac{n}{2} \right) \right)^{M-1+\frac{n}{2}}
\end{aligned}$$

Note that in (4.10) we have used the fact that the function  $\vartheta(\xi) = \xi^{M-1+\frac{n}{2}} \exp \left( \frac{M/16}{\xi} \right)$  achieves its minimum on  $\mathbb{R}_+$  at  $\xi = \frac{M/16}{M-1+\frac{n}{2}}$ .

On the other hand, for any  $(y, s) \in B_{\frac{1}{4}} \times [\frac{-1}{8M}, 0)$ , the parabolic cylinder  $Q(y, s; \sqrt{-s}) = B_{\sqrt{-s}}^n(y) \times (2s, s)$  is contained in  $B_{1/2}^n \times (\frac{-1}{M}, 0)$  and hence the LHS of (4.8) is bounded below by

$$(4.11) \quad \int_{B_{1/2}^n \times (\frac{-1}{M}, 0)} v^2 \varphi^{-M} \Phi dx dt \geq \frac{\exp \frac{-1/4}{-8s}}{(4\pi)^{\frac{n}{2}} (-2s)^{M+\frac{n}{2}}} \int_{Q(y, s; \sqrt{-s})} v^2 dx dt$$

Combining (4.8), (4.9), (4.10), (4.11), we conclude that for  $(y, s) \in Q(0, 0; \frac{-1}{8M})$ ,

$$(4.12) \quad \int_{Q(y, s; \sqrt{-s})} v^2 dx dt$$



$$\leq C(n, \lambda, L, \sigma) \left( \frac{64\sigma}{e} (-sM) \right)^{M-1+\frac{n}{2}} \left( \|\partial_x v\|_{L^\infty(B_1 \times [-T, 0])}^2 + \|v\|_{L^\infty(B_1 \times [-T, 0])}^2 \right)$$

Now let  $\beta = \frac{1}{2} \left( \frac{64\sigma}{e} \right)^{-1}$ . For each  $(y, s) \in B_{1/4}^n \times [\frac{-\beta}{4L^2(n+\sigma)}T, 0)$ , we choose  $M = \frac{\beta}{-s}$  so that  $M \geq \frac{4L^2(n+\sigma)}{T}$  (and note that  $\frac{-1}{8M} \leq s < 0$ ). By (4.12), we get

$$(4.13) \quad \int_{Q(y, s; \sqrt{-s})} |v| dx dt \leq \left( \int_{Q(y, s; \sqrt{-s})} v^2 dx dt \right)^{\frac{1}{2}}$$

$$\leq C(n, \lambda, L, \sigma) \sqrt{(-s)^{-\frac{n}{2}-1} \left( \frac{1}{2} \right)^{-\frac{\beta}{s}-1+\frac{n}{2}}} \left( \|\partial_x v\|_{L^\infty(B_1 \times [-T, 0])} + \|v\|_{L^\infty(B_1 \times [-T, 0])} \right)$$

$$\leq C(n, \lambda, L, \sigma) \left( 2^{\frac{\beta}{4}} \right)^{\frac{1}{s}} \left( \|\partial_x v\|_{L^\infty(B_1 \times [-T, 0])} + \|v\|_{L^\infty(B_1 \times [-T, 0])} \right)$$

Let  $\alpha = \frac{\beta}{4L^2(n+\sigma)}$ ,  $\Lambda = \max \left\{ C(n, \lambda, L, \sigma), \left( \frac{\beta}{4} \ln 2 \right)^{-1} \right\}$ , then (4.4) follows from (4.13) and Lemma 4.1.  $\square$

Combining Proposition 3.7, Proposition 3.8 with Proposition 4.5, we could show the exponential decay of  $h_t = h(\cdot, t)$  as  $t \nearrow 0$  as in [W] (see also [N]).

**Proposition 4.7** (Exponential decay of the deviation). *There exist  $R = R(\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \|F\|_{C^3(U)}, \lambda, \varkappa) \geq 1$ ,  $\Lambda = \Lambda(n, \mathcal{C}, \|F\|_{C^3(U)}, \lambda) > 0$ ,  $\alpha = \alpha(n, \mathcal{C}, \|F\|_{C^3(U)}, \lambda) \in (0, 1)$  such that for  $X \in \Sigma_t \setminus \bar{B}_R$ ,  $t \in [-\alpha, 0)$ , there holds*

$$|\nabla_{\Sigma_t} h| + |h| \leq \Lambda \exp \left( \frac{|X|^2}{\Lambda t} \right)$$

*Proof.* Fix  $\hat{X} \in \Sigma \setminus \bar{B}_R$  with  $R = R(\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \|F\|_{C^3(U)}, \lambda, \varkappa) \geq 1$ , we would first show that near  $\hat{X}$ , there is a “normal parametrization” for the flow  $\{\Sigma_t\}$  for  $t \in [-1, 0]$ .

Recall that at the beginning of Section 3, we show that there exists an uniform radius  $\rho = \rho(n, \mathcal{C}) \in (0, 1)$  so that near  $\hat{X}$ , each  $\Sigma_t$  is the graph of the function  $u_t = u(\cdot, t)$  defined on  $B_{\rho|\hat{X}|}^n \subset T_{\hat{X}_C} \mathcal{C}$  for  $t \in [-1, 0]$ , where  $\hat{X}_C = \Pi(\hat{X})$  is the normal projection of  $\hat{X}$  onto  $\mathcal{C}$ . Note also that  $|\hat{X}_C|$  is comparable with  $|\hat{X}|$ . In other words, locally near  $\hat{X}$ , we have the following “vertical parametrization” of the flow  $\{\Sigma_t\}_{-1 \leq t \leq 0}$ :

$$X = X(x, t) \equiv \hat{X}_C + (x, u(x, t))$$

Here we assume that the unit-normal of  $\mathcal{C}$  at  $\hat{X}_C$  to be  $(0, 1)$  for ease of notation. For this vertical parametrization, we may decompose the velocity vector into the normal part and the tangential part as follows:

$$\partial_t X = F(A^\#(x, t)) N(x, t) + \sum_{i=1}^n \frac{\partial_i u \partial_t u}{1 + |\partial_x u|^2} \partial_i X$$

where  $A^\#(x, t)$ ,  $N(x, t)$  are the shape operator and the unit-normal of  $\Sigma_t$  at  $X(x, t)$ , respectively. Note that the normal part is from Definition 2.4.

Next, we would do suitable change of variables to go from this “vertical parametrization” to the “normal parametrization” of the flow (see Definition 3.6). For that purpose, we use the same trick as in Proposition 3.7.

Let  $x = \phi_t(\xi)$  with  $\phi_{-1} = \text{id}$  to be the local diffeomorphism on  $\Sigma_t$  generated by the following tangent vector field:

$$\mathcal{V}(x, t) = - \sum_{i=1}^n \frac{\partial_i \mathbf{u} \partial_t \mathbf{u}}{1 + |\partial_x \mathbf{u}|^2} \partial_i X \equiv - \sum_{i=1}^n \mathcal{V}^i(x, t) \partial_i X$$

That is,  $\phi_t(\xi) = \phi(\xi, t)$  satisfies

$$(4.14) \quad \partial_t \phi_t = (\mathcal{V}^1(\phi_t, t), \dots, \mathcal{V}^n(\phi_t, t)), \quad \phi_{-1}(\xi) = \xi$$

in which, by (3.4) and (3.9), we have

$$(4.15) \quad |\mathcal{V}^i| \leq C(n, \mathcal{C}, \|F\|_{C^3(U)}) |\hat{X}|^{-1} \quad \forall i = 1, \dots, n$$

Thus, by taking  $R$  sufficiently large,  $\phi_t$  is well-defined for  $\xi \in B_{\frac{R}{2}|\hat{X}|}^n$ ,  $t \in [-1, 0]$ . It follows that the reparametrization  $X = X(\phi_t(\xi), t)$  of the flow becomes a “normal parametrization” near  $\hat{X}$  for  $t \in [-1, 0]$ ; that is,

$$\frac{\partial}{\partial t} (X(\phi_t(\xi), t)) = F(A^\#(\phi_t(\xi), t)) N(\phi_t(\xi), t)$$

Let  $g_{ij}(\xi, t) = \partial_{\xi_i}(X(\phi_t(\xi), t)) \cdot \partial_{\xi_j}(X(\phi_t(\xi), t))$  be the pull-back metric associated with this “normal parametrization”, then by the evolution equation for the metric in [A], the homogeneity of  $F$  and the condition that  $\phi_{-1} = \text{id}$ , we have

$$(4.16) \quad \begin{aligned} \partial_t g_{ij}(\xi, t) &= -2F(A^\#(\phi_t(\xi), t)) A_{ij}(\phi_t(\xi), t) \\ &= -2 \left| X(\phi_t(\xi), t) \right|^{-1} F \left( \left| X(\phi_t(\xi), t) \right| A^\#(\phi_t(\xi), t) \right) A_{ij}(\phi_t(\xi), t) \end{aligned}$$

$$(4.17) \quad g_{ij}(\xi, -1) = \delta_{ij} + \partial_i \mathbf{u}(\xi, -1) \partial_j \mathbf{u}(\xi, -1)$$

where the second fundamental form  $A_t(x) \sim A_{ij}(x, t)$  is equal to

$$(4.18) \quad A_{ij}(x, t) = \frac{\partial_{ij}^2 \mathbf{u}(x, t)}{\sqrt{1 + |\partial_x \mathbf{u}(x, t)|^2}}$$

By (4.18), (3.1), (3.2), (3.3), (3.12) and the comparability of  $|X(x, t)|$  and  $|\hat{X}|$ , the  $\ell^2$  norm of the matrix  $\partial_t g_{ij}(\xi, t)$  satisfies

$$(4.19) \quad |\partial_t g_{ij}(\xi, t)| \leq C(n, \mathcal{C}, \|F\|_{C^1(U)}) |\hat{X}|^{-2}$$

So by (4.17), (3.1), (3.3) and (4.19), the pull-back metric  $g_{ij}(\xi, t)$  is equivalent to the dot product  $\delta_{ij}$ .

Let  $\Gamma_{ij}^k(\xi, t)$  be the Christoffel symbols associated with the metric  $g_{ij}(\xi, t)$ , then we have

$$(4.20) \quad \partial_t \Gamma_{ij}^k = \frac{1}{2} g^{kl} \left( \nabla_i \dot{g}_{lj} + \nabla_j \dot{g}_{il} - \nabla_l \dot{g}_{ij} \right)$$

$$(4.21) \quad \Gamma_{ij}^k(\xi, -1) = \frac{\partial_k \mathbf{u}(\xi, -1) \partial_{ij}^2 \mathbf{u}(\xi, -1)}{1 + |\partial_x \mathbf{u}(\xi, -1)|^2}$$

where

$$\dot{g}_{ij} = \partial_t g_{ij} = -2F(A^\#) A_{ij}$$

Similarly, and also by (3.15), the homogeneity of the derivative of  $F$ , the equivalence of  $g_{ij}$  and  $\delta_{ij}$ , we have

$$\begin{aligned} |\partial_t \Gamma_{ij}^k| &\leq C(n, \mathcal{C}, \|F\|_{C^1(U)}) |\hat{X}|^{-3} \\ |\Gamma_{ij}^k(\xi, -1)| &\leq C(n, \mathcal{C}, \|F\|_{C^1(U)}) |\hat{X}|^{-1} \end{aligned}$$

which implies

$$(4.22) \quad |\Gamma_{ij}^k(\xi, t)| \leq C(n, \mathcal{C}, \|F\|_{C^1(U)}) |\hat{X}|^{-1}$$

Now consider the deviation  $h$  in the local coordinates  $(\xi, t)$ , then the equation in Proposition 3.7 becomes

$$\begin{aligned} (4.23) \quad &\left| \partial_t h - \left\{ \partial_{\xi_i} (\mathbf{a}^{ij}(\xi, t) \partial_{\xi_j} h) + \Gamma_{ik}^i(\xi, t) \mathbf{a}^{kj}(\xi, t) \partial_{\xi_j} h \right\} \right| \\ &\leq C(n, \mathcal{C}, \|F\|_{C^3(U)}) \left( |\hat{X}|^{-1} |\partial_{\xi} h| + |\hat{X}|^{-2} |h| \right) \\ &\quad h(\xi, 0) = 0 \end{aligned}$$

where  $\mathbf{a}^{ij}(\xi, t) = \mathbf{a}^{ji}(\xi, t)$  satisfies (by Proposition 3.8 and (4.22))

$$(4.24) \quad \frac{\lambda}{C(n, \mathcal{C}, \|F\|_{C^3(U)})} \delta^{ij} \leq \frac{\lambda}{3} g^{ij}(\xi, t) \leq \mathbf{a}^{ij}(\xi, t) \leq \frac{3}{\lambda} g^{ij}(\xi, t) \leq \frac{C(n, \mathcal{C}, \|F\|_{C^3(U)})}{\lambda} \delta^{ij}$$

$$(4.25) \quad |\hat{X}| \left| \partial_{\xi} \mathbf{a}^{ij}(\xi, t) \right| + |\hat{X}|^2 \left| \partial_t \mathbf{a}^{ij}(\xi, t) \right| \leq C(n, \mathcal{C}, \|F\|_{C^3(U)}, \lambda)$$

Thus, by (4.22), (4.24), (4.17) and (4.19), the equation (4.23) is equivalent to

$$\begin{aligned} (4.26) \quad &\left| \partial_t h - \partial_{\xi_i} (\mathbf{a}^{ij}(\xi, t) \partial_{\xi_j} h) \right| \\ &\leq C(n, \mathcal{C}, \|F\|_{C^3(U)}, \lambda) \left( |\hat{X}|^{-1} |\partial_{\xi} h| + |\hat{X}|^{-2} |h| \right) \end{aligned}$$

for  $(\xi, t) \in B_{\frac{n}{2}|\hat{X}|}^n \times [-1, 0]$

$$h(\xi, 0) = 0$$

Let's consider the following change of variables:

$$(\xi, t) = \Xi(\bar{\xi}, \bar{t}) \equiv \left( \left( \frac{\rho}{2} |\hat{X}| \right) \bar{\xi}, \left( \frac{\rho}{2} |\hat{X}| \right)^2 \bar{t} \right)$$

and let  $\bar{h} = h \circ \Xi$ ,  $\bar{\mathbf{a}}^{ij} = \mathbf{a}^{ij} \circ \Xi$ . Then (4.26) in the new variables becomes

$$\begin{aligned} (4.27) \quad &\left| \partial_{\bar{t}} \bar{h} - \partial_{\bar{\xi}_i} \left( \bar{\mathbf{a}}^{ij}(\bar{\xi}, \bar{t}) \partial_{\bar{\xi}_j} \bar{h} \right) \right| \leq C(n, \mathcal{C}, \|F\|_{C^3(U)}, \lambda, \rho) (|\partial_{\bar{\xi}} \bar{h}| + |\bar{h}|) \\ &\quad \bar{h} \Big|_{\bar{t}=0} = 0 \end{aligned}$$

and (4.24), (4.25) are translated into

$$(4.28) \quad \frac{\lambda}{C(n, \mathcal{C}, \|F\|_{C^3(U)})} \delta^{ij} \leq \bar{\mathbf{a}}^{ij}(\bar{\xi}, \bar{t}) \leq \frac{C(n, \mathcal{C}, \|F\|_{C^3(U)})}{\lambda} \delta^{ij}$$

$$(4.29) \quad \left| \partial_{\bar{\xi}} \bar{\mathbf{a}}^{ij}(\bar{\xi}, \bar{t}) \right| + \left| \partial_{\bar{t}} \bar{\mathbf{a}}^{ij}(\bar{\xi}, \bar{t}) \right| \leq C(n, \mathcal{C}, \|F\|_{C^3(U)}, \lambda, \rho)$$

for  $\bar{\xi} \in B_1^n$ ,  $\bar{t} \in \left[ -\left( \frac{\rho}{2} |\hat{X}| \right)^{-2}, 0 \right]$ .

Applying Proposition 4.5 to  $\bar{h}(\bar{\xi}, \bar{t})$ , we may conclude that there exist  $\tilde{\Lambda} = \tilde{\Lambda}(n, \mathcal{C}, \|F\|_{C^3(U)}, \lambda) > 0$ ,  $\alpha = \alpha(n, \mathcal{C}, \|F\|_{C^3(U)}, \lambda) \in (0, 1)$  for which the following holds:

$$(4.30) \quad |\partial_{\bar{\xi}} \bar{h}| + |\bar{h}| \leq \tilde{\Lambda} \exp\left(\frac{1}{\tilde{\Lambda} \bar{t}}\right) \left( \|\partial_{\bar{\xi}} \bar{h}\|_{L^\infty(B_1^n \times [-(\frac{\rho}{2}|\hat{X}|)^{-2}, 0])} + \|\bar{h}\|_{L^\infty(B_1^n \times [-(\frac{\rho}{2}|\hat{X}|)^{-2}, 0])} \right)$$

for  $(\bar{\xi}, \bar{t}) \in B_{1/4}^n \times [-\alpha(\frac{\rho}{2}|\hat{X}|)^{-2}, 0)$ .

Undoing change of variables, (4.30) becomes

$$(4.31) \quad \frac{\rho}{2} |\hat{X}| |\partial_{\xi} h| + |h| \leq \tilde{\Lambda} \exp\left(\frac{|\hat{X}|^2}{\tilde{\Lambda} t}\right) \left( \frac{\rho}{2} |\hat{X}| \|\partial_{\xi} h\|_{L^\infty(B_{\frac{\rho}{2}|\hat{X}|}^n \times [-1, 0])} + \|h\|_{L^\infty(B_{\frac{\rho}{2}|\hat{X}|}^n \times [-1, 0])} \right)$$

for  $(\xi, t) \in B_{\frac{\rho}{2}|\hat{X}|}^n \times [-\alpha, 0)$ .

Note that the pull-back metric  $g_{ij}(\xi, t)$  is equivalent to the dot product  $\delta_{ij}$  and that  $|X(x, t)|$  is comparable with  $|\hat{X}|$ . The conclusion follows immediately.  $\square$

Next, we would go from the exponential decay to identically vanishing of the deviation  $h$  outside a compact set. To this end, we have to derive a different type of Carleman's inequality on the flow  $\{\Sigma_t\}_{-1 \leq t \leq 0}$ , which is done through two lemmas. The first lemma is a modification of the integral equality in [EF].

**Lemma 4.8.** *Let  $(M, g_t)$  be a flow of Riemannian manifolds and  $P$  be a differential operator on the flow defined by*

$$Pv = \partial_t v - \nabla_{g_t} \cdot (a_t dv) = \partial_t v - \nabla_i (a^{ij}(\cdot, t) \nabla_j v)$$

where  $a_t = a(\cdot, t)$  is a symmetric 2-tensor on  $(M, g_t)$ . Then given functions  $G, \Psi \in C^{2,1}(M \times [-T, 0])$  with  $G > 0$ , define a function  $\Phi$  as

$$(4.32) \quad \Phi = \frac{\partial_t G + \nabla_i (a^{ij} \nabla_j G) + \frac{1}{2} \text{tr}(\partial_t g) G}{G} \\ = \partial_t \ln G + \nabla_i (a^{ij} \nabla_j \ln G) + a^{ij} \nabla_i \ln G \nabla_j \ln G + \frac{1}{2} \text{tr}(\partial_t g)$$

and a 2-tensor  $\Upsilon$  as

$$(4.33) \quad \Upsilon^{ij} = a^{ik} a^{jl} \nabla_{kl}^2 \ln G - \frac{1}{2} \partial_t a^{ij} \\ + \frac{1}{2} (a^{ik} \nabla_k a^{jl} + a^{jk} \nabla_k a^{il} - a^{lk} \nabla_k a^{ij}) \nabla_l \ln G$$

It follows that for any  $u \in C_c^{2,1}(M \times [-T, 0])$ , there holds

$$(4.34) \quad \int_M \left\{ (2\Upsilon^{ij} - (\Phi - \Psi) a^{ij}) \nabla_i u \nabla_j u + \frac{1}{2} (\partial_t \Psi - \nabla_i (a^{ij} \nabla_j \Psi) + (\Phi - \Psi) \Psi) u^2 \right\} G d\mu_t \\ = \left\{ \int_M 2Pu \left( \partial_t u + a^{ij} \nabla_i \ln G \nabla_j u + \frac{1}{2} \Psi u \right) G d\mu_t \right. \\ \left. - \int_M 2 \left( \partial_t u + a^{ij} \nabla_i \ln G \nabla_j u + \frac{1}{2} \Psi u \right)^2 G d\mu_t \right\}$$

$$-\partial_t \left\{ \int_M \left( a^{ij} \nabla_i u \nabla_j u - \frac{1}{2} \Psi u^2 \right) G d\mu_t \right\}$$

where  $\mu_t$  is the volume form of  $(M, \mathbf{g}_t)$ .

*Proof.* Let's begin with

$$(4.35) \quad \begin{aligned} & \partial_t \left\{ \int_M a^{ij} \nabla_i u \nabla_j u G d\mu_t \right\} \\ &= \int_M \left\{ 2a^{ij} \nabla_j u \nabla_i \partial_t u G + a^{ij} \nabla_i u \nabla_j u \left( \partial_t G + \frac{1}{2} \text{tr}(\partial_t g) G \right) + \partial_t a^{ij} \nabla_i u \nabla_j u G \right\} d\mu_t \end{aligned}$$

in which we use the commutativity:

$$\partial_t du = d \partial_t u, \quad \text{where} \quad du \sim \nabla_i u$$

and the evolution equation of the volume form:

$$(4.36) \quad \partial_t d\mu_t = \frac{1}{2} \text{tr}(\partial_t g) d\mu_t$$

Applying integration by parts on  $(M, \mathbf{g}_t)$ , (4.35) becomes

$$(4.37) \quad \begin{aligned} & \left\{ \int_M -2(\nabla_i (a^{ij} \nabla_j u) + a^{ij} \nabla_i \ln G \nabla_j u) \partial_t u G d\mu_t + \int_M a^{ij} \nabla_i u \nabla_j u \left( \partial_t G + \nabla_k (a^{kl} \nabla_l G) + \frac{1}{2} \text{tr}(\partial_t g) G \right) d\mu_t \right. \\ & \left. - \int_M a^{ij} \nabla_i u \nabla_j u \nabla_k (a^{kl} \nabla_l G) d\mu_t + \int_M \partial_t a^{ij} \nabla_i u \nabla_j u G d\mu_t \right\} \end{aligned}$$

By (4.32), integration by parts twice and the symmetry of  $a_t$ , (4.37) becomes

$$(4.38) \quad \begin{aligned} & \left\{ -2 \int_M (\nabla_i (a^{ij} \nabla_j u) + a^{ij} \nabla_i \ln G \nabla_j u) \partial_t u G d\mu_t + \int_M a^{ij} \nabla_i u \nabla_j u \Phi G d\mu_t \right. \\ & + \int_M \{ \nabla_k a^{ij} \nabla_i u \nabla_j u a^{kl} \nabla_l \ln G - 2 \nabla_j (a^{ij} \nabla_i u) \nabla_k u a^{kl} \nabla_l \ln G - 2 a^{ij} \nabla_i u \nabla_k u \nabla_j a^{kl} \nabla_l \ln G \} G d\mu_t \\ & \left. - 2 \int_M a^{ij} \nabla_i u \nabla_k u a^{kl} \nabla_{jl}^2 G d\mu_t + \int_M \partial_t a^{ij} \nabla_i u \nabla_j u G d\mu_t \right\} \end{aligned}$$

Then we reorganize (4.38) (in order to make up the term  $P$ ) to get

$$(4.39) \quad \begin{aligned} & \left\{ 2 \int_M \left\{ (\partial_t u - \nabla_i (a^{ij} \nabla_j u)) (\partial_t u + a^{kl} \nabla_k \ln G \nabla_l u) - (\partial_t u)^2 - 2 a^{ij} \nabla_i \ln G \nabla_j u \partial_t u \right\} G d\mu_t \right. \\ & + \int_M \Phi a^{ij} \nabla_i u \nabla_j u G d\mu_t - 2 \int_M a^{ij} a^{kl} (\nabla_{jl}^2 \ln G + \nabla_j \ln G \nabla_l \ln G) \nabla_i u \nabla_k u G d\mu_t \\ & \left. + \int_M \{ a^{kl} \nabla_k a^{ij} \nabla_l \ln G \nabla_i u \nabla_j u - 2 a^{ij} \nabla_j a^{kl} \nabla_l \ln G \nabla_i u \nabla_k u + \partial_t a^{ij} \nabla_i u \nabla_j u \} G d\mu_t \right\} \end{aligned}$$

By (4.33), (4.39) becomes

$$\begin{aligned} & \left\{ 2 \int_M \left\{ (\partial_t u - \nabla_i (a^{ij} \nabla_j u)) (\partial_t u + a^{kl} \nabla_k \ln G \nabla_l u) - (\partial_t u + a^{ij} \nabla_i \ln G \nabla_j u)^2 \right\} G d\mu_t \right. \\ & \quad \left. + \int_M \Phi a^{ij} \nabla_i u \nabla_j u G d\mu_t - 2 \int_M \Upsilon^{ij} \nabla_i u \nabla_j u G d\mu_t \right\} \\ &= \left\{ 2 \int_M P u \left( \partial_t u + a^{ij} \nabla_i \ln G \nabla_j u + \frac{1}{2} \Psi u \right) G d\mu_t - \int_M (\partial_t u - \nabla_i (a^{ij} \nabla_j u)) \Psi u G d\mu_t \right\} \end{aligned}$$

$$\begin{aligned}
& -2 \int_{\mathbf{M}} \left( \partial_t u + a^{ij} \nabla_i \ln G \nabla_j u + \frac{1}{2} \Psi u \right)^2 G d\mu_t + 2 \int_{\mathbf{M}} \left( \partial_t u + a^{ij} \nabla_i \ln G \nabla_j u + \frac{1}{2} \Psi u \right) \Psi u G d\mu_t \\
(4.40) \quad & - \frac{1}{2} \int_{\mathbf{M}} \Psi^2 u^2 G d\mu_t - \int_{\mathbf{M}} (2\Upsilon^{ij} - \Phi a^{ij}) \nabla_i u \nabla_j u G d\mu_t \}
\end{aligned}$$

For the second term of (4.40), by the product rule and integration by parts, we get

$$\begin{aligned}
(4.41) \quad & - \int_{\mathbf{M}} (\partial_t u - \nabla_i (a^{ij} \nabla_j u)) u \Psi G d\mu_t \\
& = -\frac{1}{2} \int_{\mathbf{M}} (\partial_t u^2 - \nabla_i (a^{ij} \nabla_j u^2) + 2a^{ij} \nabla_i u \nabla_j u) \Psi G d\mu_t \\
& = \left\{ \frac{1}{2} \int_{\mathbf{M}} \left( \partial_t \Psi G + \Psi \left( \partial_t G + \frac{1}{2} \text{tr}(\partial_t g) G \right) \right) u^2 d\mu_t - \partial_t \left( \int_{\mathbf{M}} \frac{1}{2} \Psi^2 u^2 G d\mu_t \right) \right. \\
& - \int_{\mathbf{M}} a^{ij} \nabla_i u \nabla_j u \Psi G d\mu_t + \frac{1}{2} \int_{\mathbf{M}} \{ \nabla_j (a^{ij} \nabla_i \Psi) G + 2a^{ij} \nabla_i G \nabla_j \Psi + \Psi \nabla_j (a^{ij} \nabla_i G) \} u^2 d\mu_t \\
& = \left\{ \frac{1}{2} \int_{\mathbf{M}} (\partial_t \Psi + \nabla_j (a^{ij} \nabla_i \Psi) + \Phi \Psi + a^{ij} \nabla_i \ln G \nabla_j \Psi) u^2 G d\mu_t \right. \\
& \quad \left. - \int_{\mathbf{M}} \Psi a^{ij} \nabla_i u \nabla_j u G d\mu_t - \partial_t \left( \int_{\mathbf{M}} \frac{1}{2} \Psi^2 u^2 G d\mu_t \right) \right\}
\end{aligned}$$

Likewise, for the fourth term of (4.40), we have

$$\begin{aligned}
(4.42) \quad & 2 \int_{\mathbf{M}} \left( \partial_t u + a^{ij} \nabla_i \ln G \nabla_j u + \frac{1}{2} \Psi u \right) \Psi u G d\mu_t \\
& = \int_{\mathbf{M}} \partial_t u^2 \Psi G d\mu_t + \int_{\mathbf{M}} a^{ij} \nabla_i G \nabla_j u^2 \Psi d\mu_t + \int_{\mathbf{M}} \Psi^2 u^2 G d\mu_t \\
& = \left\{ - \int_{\mathbf{M}} \left( \partial_t \Psi G + \Psi \left( \partial_t G + \frac{1}{2} \text{tr}(\partial_t g) G \right) \right) u^2 d\mu_t + \partial_t \left( \int_{\mathbf{M}} \Psi u^2 G d\mu_t \right) \right. \\
& \quad \left. + \int_{\mathbf{M}} \Psi^2 u^2 G d\mu_t - \int_{\mathbf{M}} (\nabla_j (a^{ij} \nabla_i G) \Psi + a^{ij} \nabla_i G \nabla_j \Psi) u^2 d\mu_t \right\} \\
& = - \int_{\mathbf{M}} (\partial_t \Psi + \Phi \Psi + a^{ij} \nabla_i \ln G \nabla_j \Psi - \Psi^2) u^2 G d\mu_t + \partial_t \left( \int_{\mathbf{M}} \Psi u^2 G d\mu_t \right)
\end{aligned}$$

Combining (4.40), (4.41), (4.42) to get (4.34).  $\square$

We hereafter consider the Riemannian manifold in Lemma 4.8 to be the time-slice  $\Sigma_t$  with the induced metric  $g_t$ , which evolves (in “normal parametrization”) like

$$\partial_t g = -2F(A^\#)A$$

(see [A]). The differential operator in Lemma 4.8 is taken to be the one in Proposition 3.7.

For the second lemma, we would choose suitable weight function  $G$  and auxiliary function  $\Psi$  in Lemma 4.8 in order to bound the LHS of (4.34) from below. The choice of  $G$  is due to [ESS] and [W]. As for  $\Psi$ , it is not shown in [W] but is used here in order to deal with the last term in (4.33), which comes from the nonlinear nature of  $F$ . Note that in the linear case when  $F(S) = \text{tr}(S)$  (see [W]), the coefficients of the differential operator in Proposition 3.7 becomes  $\mathbf{a}^{ij} = g^{ij}$ ; besides, (4.33) is reduced to

$$\Upsilon^{ij} = g^{ik} g^{jl} \nabla_{kl}^2 \ln G - H A^{ij}$$

The idea of using an auxiliary function for the nonlinear case is motivated by [N].

**Lemma 4.9.** *Assume that  $\varkappa \leq 6^{-4}\lambda^3$  in (2.1) and (2.2). Then there exists  $R = R(\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \|F\|_{C^3(U)}, \lambda, \varkappa) \geq 1$  so that for any constants  $M \geq 1$ ,  $\tau \in (0, 1]$ , let*

$$(4.43) \quad G = \exp \left( M(t + \tau) |X|^{\frac{3}{2}} + |X|^2 \right)$$

$$(4.44) \quad \begin{aligned} \Psi = & \left\{ \left( \frac{3}{2} M(t + \tau) |X|^{-\frac{1}{2}} + 2 \right)^2 \mathbf{a}^{ij} (X \cdot \partial_i X) (X \cdot \partial_j X) + M |X|^{\frac{3}{2}} \right. \\ & + \frac{1}{2} \left( \frac{3}{2} M(t + \tau) |X|^{-\frac{1}{2}} + 2 \right) \left( \text{tr}(\mathbf{a}) - \frac{\lambda}{3} \right) \\ & \left. + \left( \text{tr}(\mathbf{a}) - \frac{\lambda}{3} \right) + \frac{3}{4} M(t + \tau) |X|^{-\frac{5}{2}} (\text{tr}(\mathbf{a}) |X|^2 - \mathbf{a}^{ij} (X \cdot \partial_i X) (X \cdot \partial_j X)) \right\} \end{aligned}$$

(note that  $G > 0$  and  $\Psi \geq 0$ ), there hold

$$(4.45) \quad 2\Upsilon^{ij} - (\Phi - \Psi) \mathbf{a}^{ij} \geq \frac{\lambda^2}{9} g^{ij}$$

$$(4.46) \quad \frac{1}{2} (\partial_t \Psi - \nabla_i (\mathbf{a}^{ij} \nabla_j \Psi) + (\Phi - \Psi) \Psi) \geq \frac{\lambda^2}{9} |X|^2$$

for  $X \in \Sigma_t \setminus \bar{B}_R$ ,  $t \in [-\tau, 0)$ , where  $\text{tr}(\mathbf{a}) = g_{ij} \mathbf{a}^{ij}$ ,  $\Phi$  and  $\Upsilon^{ij}$  are defined in (4.32) and (4.33), respectively, with the covariant derivative is taken w.r.t  $\Sigma_t$ ,  $\partial_t g = -2F(A^\#)A$ , and  $a^{ij} = \mathbf{a}^{ij}$ .

*Remark 4.10.* In view of (3.53), the hypothesis that  $\varkappa \leq 6^{-4}\lambda^3$  amounts to requiring the smallness of  $|X| |\nabla_{\Sigma_t} \mathbf{a}|$  (compared with the ellipticity of  $\mathbf{a}$ ). Similar hypothesis also appears in [N] and [WZ] when using Carleman's inequalities to prove the backward uniqueness of parabolic equations.

*Proof.* Let's start with computing the covariant derivatives of  $\ln G$ :

$$(4.47) \quad \nabla_i \ln G = \left( \frac{3}{2} M(t + \tau) |X|^{-\frac{1}{2}} + 2 \right) (X \cdot \partial_i X)$$

$$(4.48) \quad \begin{aligned} \nabla_{ij}^2 \ln G = & \left\{ \left( \frac{3}{2} M(t + \tau) |X|^{-\frac{1}{2}} + 2 \right) (g_{ij} + X \cdot N A_{ij}) \right. \\ & - \frac{3}{4} M(t + \tau) |X|^{-\frac{5}{2}} (|X|^2 g_{ij} - (X \cdot \partial_i X) (X \cdot \partial_j X)) \\ & \left. + 2t \left( \frac{3}{2} M(t + \tau) |X|^{-\frac{1}{2}} + 2 \right) F(A^\#) A_{ij} \right\} \end{aligned}$$

and its evolution

$$(4.49) \quad \begin{aligned} \partial_t \ln G = & M |X|^{\frac{3}{2}} + \left( \frac{3}{2} M(t + \tau) |X|^{-\frac{1}{2}} + 2 \right) (X \cdot \partial_t X) \\ = & M |X|^{\frac{3}{2}} + 2t \left( \frac{3}{2} M(t + \tau) |X|^{-\frac{1}{2}} + 2 \right) F(A^\#)^2 \end{aligned}$$

in which we use the  $F$  curvature flow equation in normal parametrization (see Definition 3.6):

$$\partial_t X = F(A^\#) N$$

and the  $F$  self-shrinker equation for  $\Sigma_t = \sqrt{-t}\Sigma$  (in Definition 2.4):

$$X \cdot N = 2tF(A^\#)$$

Thus, by (4.32), (4.47), (4.48) and (4.49), we have

$$(4.50) \quad \begin{aligned} \Phi = & \left\{ \left( \frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}} + 2 \right)^2 \mathbf{a}^{ij}(X \cdot \partial_i X)(X \cdot \partial_j X) + M|X|^{\frac{3}{2}} \right. \\ & + \frac{1}{2} \left( \frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}} + 2 \right) \text{tr}(\mathbf{a}) \\ & + \text{tr}(\mathbf{a}) + \frac{3}{4}M(t+\tau)|X|^{-\frac{5}{2}} (\text{tr}(\mathbf{a})|X|^2 - \mathbf{a}^{ij}(X \cdot \partial_i X)(X \cdot \partial_j X)) \\ & \left. + \left( \frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}} + 2 \right) \{ (\nabla_i \mathbf{a}^{ij})(X \cdot \partial_j X) + 2tF(A^\#)(F(A^\#) + \mathbf{a}^{ij}A_{ij}) \} - F(A^\#)H \right\} \end{aligned}$$

which, together with (4.44), implies that

$$(4.51) \quad \begin{aligned} \Phi - \Psi = & \left\{ \frac{\lambda}{2} \left( \frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}} + 2 \right) + \frac{\lambda}{3} \right. \\ & \left. + \left( \frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}} + 2 \right) \{ (\nabla_k \mathbf{a}^{kl})(X \cdot \partial_l X) + 2tF(A^\#)(F(A^\#) + \mathbf{a}^{kl}A_{kl}) \} - F(A^\#)H \right\} \end{aligned}$$

By (4.33), (4.47), (4.48) and (4.51),

$$(4.52) \quad \begin{aligned} 2\Upsilon^{ij} - (\Phi - \Psi)\mathbf{a}^{ij} = & \left\{ \left( \frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}} + 2 \right) \left( \mathbf{a}^{ik}\mathbf{a}^{jl}g_{kl} - \frac{\lambda}{6}\mathbf{a}^{ij} \right) \right. \\ & + \left( 2\mathbf{a}^{ik}\mathbf{a}^{jl}g_{kl} - \frac{\lambda}{3}\mathbf{a}^{ij} \right) + \frac{3}{2}M(t+\tau)|X|^{-\frac{5}{2}}\mathbf{a}^{ik}\mathbf{a}^{jl}(|X|^2g_{kl} - (X \cdot \partial_k X)(X \cdot \partial_l X)) \\ & + \left( \frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}} + 2 \right) \{ \mathbf{a}^{ik}\nabla_k \mathbf{a}^{jl} + \mathbf{a}^{jk}\nabla_k \mathbf{a}^{il} - \mathbf{a}^{lk}\nabla_k \mathbf{a}^{ij} - \mathbf{a}^{ij}\nabla_k \mathbf{a}^{kl} \} (X \cdot \partial_l X) \\ & + \left( \frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}} + 2 \right) (2\mathbf{a}^{ik}\mathbf{a}^{jl}A_{kl} - \mathbf{a}^{ij}\mathbf{a}^{kl}A_{kl} - F(A^\#)\mathbf{a}^{ij})2tF(A^\#) \\ & \left. - \partial_t \mathbf{a}^{ij} + F(A^\#)H\mathbf{a}^{ij} \right\} \end{aligned}$$

which can be estimated from below (using (3.52), (3.53), (3.55), (3.12), (3.15) and the homogeneity of  $F$ ) by

$$(4.53) \quad \begin{aligned} 2\Upsilon^{ij} - (\Phi - \Psi)\mathbf{a}^{ij} \geq & \left\{ \left( \frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}} + 2 \right) \left( \left( \frac{\lambda^2}{18} - 36\frac{\varkappa}{\lambda} \right) g^{ij} + O(|X|^{-2}) \right) \right. \\ & \left. + \frac{\lambda^2}{9}g^{ij} + O(|X|^{-2}) \right\} \end{aligned}$$

where the notation  $O(|X|^{-2})$  means that

$$\left| O(|X|^{-2}) \right| \leq C(n, \mathcal{C}, \|F\|_{C^3(U)})|X|^{-2}$$

Then (4.45) follows from (4.43) and the hypothesis ( $\varkappa \leq 6^{-4}\lambda^3$ ) provided that  $R \gg 1$  (independent of  $M$  and  $\tau$ ).



On the other hand, by (3.52), (3.53), (3.12), (3.15), the homogeneity of  $F$ , the hypothesis that  $\varkappa \leq 6^{-4}\lambda^3$  (note that  $\lambda \in (0, 1]$ ) and  $R \gg 1$  (independent of  $M$  and  $\tau$ ), we can estimate (4.51) from below by

$$\begin{aligned} \Phi - \Psi &\geq \left( \frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}} + 2 \right) \left( \frac{\lambda}{6} - 3\varkappa + O(|X|^{-2}) \right) + \frac{\lambda}{3} + O(|X|^{-2}) \\ (4.54) \quad &\geq \left( \frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}} + 2 \right) \frac{\lambda}{9} + \frac{\lambda}{6} \end{aligned}$$

Similarly, from the  $F$  self-shrinker equation for  $\Sigma_t$ , the tangential component of the position vector can be estimated by

$$\begin{aligned} (4.55) \quad |X^\top|^2 &= |X|^2 - (X \cdot N)^2 = |X|^2 - (2tF(A^\#))^2 \\ &= |X|^2 - (2tF(|X|A^\#))^2 |X|^{-2} = |X|^2 + O(|X|^{-2}) \end{aligned}$$

Consequently, (4.44) can be estimated from below, using (3.52) and (4.55), by

$$\begin{aligned} (4.56) \quad \Psi &\geq \left( \frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}} + 2 \right)^2 \mathbf{a}^{ij} (X \cdot \partial_i X) (X \cdot \partial_j X) + M|X|^{\frac{3}{2}} \\ &\geq \left( \frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}} + 2 \right)^2 \left( \frac{\lambda}{3}|X|^2 + O(|X|^{-2}) \right) + M|X|^{\frac{3}{2}} \end{aligned}$$

Multiply (4.54) and (4.56) to get

$$\begin{aligned} (4.57) \quad (\Phi - \Psi)\Psi &\geq \left\{ \left( \frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}} + 2 \right)^3 \frac{1}{36}\lambda^2|X|^2 \right. \\ &\quad \left. + \left( \frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}} + 2 \right)^2 \frac{1}{27}\lambda^2|X|^2 + \left( \frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}} + 2 \right) \frac{\lambda}{9}M|X|^{\frac{3}{2}} + \frac{\lambda}{6}M|X|^{\frac{3}{2}} \right\} \end{aligned}$$

To achieve (4.46), let's first rearrange (4.44) to get

$$\begin{aligned} (4.58) \quad \Psi &= \left\{ \left( \frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}} + 2 \right)^2 \mathbf{a}^{kl} (X \cdot \partial_k X) (X \cdot \partial_l X) + M|X|^{\frac{3}{2}} \right. \\ &\quad \left. + \left( \frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}} + 2 \right) \left( \operatorname{tr}(\mathbf{a}) - \frac{\mathbf{a}^{kl} (X \cdot \partial_k X) (X \cdot \partial_l X)}{2|X|^2} - \frac{\lambda}{6} \right) \right. \\ &\quad \left. + \frac{\mathbf{a}^{kl} (X \cdot \partial_k X) (X \cdot \partial_l X)}{|X|^2} - \frac{\lambda}{3} \right\} \end{aligned}$$

Then we take time-derivative of (4.58) and estimate the result by Proposition 3.8, (3.12), (3.15), the homogeneity of  $F$  and its derivatives, the  $F$  self-shrinker equation for  $\Sigma_t$  (i.e.  $X \cdot N = 2tF(A^\#)$ ) and the  $F$  curvature flow equation (i.e.  $\partial_t X = F(A^\#)N$ ), and also assuming that  $R \gg 1$  (depending on  $\lambda$ ). Note that we could simplify the computation by taking “normal coordinates” of  $\Sigma_t$ . For instance, let's compute and estimate the time-derivative of the first term in (4.58) as follows:

$$\begin{aligned} (4.59) \quad &\partial_t \left\{ \left( \frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}} + 2 \right)^2 \mathbf{a}^{kl} (X \cdot \partial_k X) (X \cdot \partial_l X) \right\} \\ &= \left\{ 2 \left( \frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}} + 2 \right) \left\{ \frac{3}{2}M|X|^{-\frac{1}{2}} + \frac{3}{2}M(t+\tau) \left( -\frac{1}{2}|X|^{-\frac{3}{2}} \right) \frac{X \cdot F(A^\#)N}{|X|} \right\} \mathbf{a}^{kl} (X \cdot \partial_k X) (X \cdot \partial_l X) \right. \end{aligned}$$

$$+ \left( \frac{3}{2} M(t + \tau) |X|^{-\frac{1}{2}} + 2 \right)^2 \{ (\partial_t \mathbf{a}^{kl}) (X \cdot \partial_k X) (X \cdot \partial_l X) + 2 \mathbf{a}^{kl} (X \cdot \partial_k X) (X \cdot \partial_l (F(A^\#) N)) \}$$

By taking normal coordinates, we may assume (at the point of consideration) that  $g_{ij} = \delta_{ij}$  (so the norm in Proposition 3.8 becomes  $\ell^2$  norm),  $\{\partial_1 X, \dots, \partial_n X, N\}$  is an orthonormal basis for  $\mathbb{R}^{n+1}$ , and the last term in (4.59) can be computed and estimated by

$$\begin{aligned} \partial_l (F(A^\#) N) &= \frac{\partial F}{\partial S_i^j} (A^\#) \left( \partial_l A_i^j \right) N + F(A^\#) (-A_l^k \partial_k X) \\ &= \frac{\partial F}{\partial S_i^j} (|X| A^\#) \left( \nabla_l A_i^j \right) N + |X|^{-1} F(|X| A^\#) (-A_l^k \partial_k X) = O(|X|^{-2}) \end{aligned}$$

Thus, (4.59) can be estimated by

$$\begin{aligned} &\left( \frac{3}{2} M(t + \tau) |X|^{-\frac{1}{2}} + 2 \right) \left( 3M|X|^{-\frac{1}{2}} + M \cdot O(|X|^{-\frac{9}{2}}) \right) \mathbf{a}^{kl} (X \cdot \partial_k X) (X \cdot \partial_l X) \\ &\quad + \left( \frac{3}{2} M(t + \tau) |X|^{-\frac{1}{2}} + 2 \right)^2 O(1) \end{aligned}$$

Doing the same thing to other terms in (4.58) leads to

$$\begin{aligned} (4.60) \quad \partial_t \Psi &= \left\{ \left( \frac{3}{2} M(t + \tau) |X|^{-\frac{1}{2}} + 2 \right) \left( 3M|X|^{-\frac{1}{2}} + M \cdot O(|X|^{-\frac{9}{2}}) \right) \mathbf{a}^{kl} (X \cdot \partial_k X) (X \cdot \partial_l X) \right. \\ &\quad \left. + \left( \frac{3}{2} M(t + \tau) |X|^{-\frac{1}{2}} + 2 \right)^2 O(1) + M \cdot O(|X|^{-\frac{1}{2}}) + \left( \frac{3}{2} M(t + \tau) |X|^{-\frac{1}{2}} + 2 \right) O(|X|^{-2}) + O(|X|^{-2}) \right\} \\ &\geq \left( \frac{3}{2} M(t + \tau) |X|^{-\frac{1}{2}} + 2 \right) \left( \frac{2}{3} \lambda M |X|^{\frac{3}{2}} \right) + \left( \frac{3}{2} M(t + \tau) |X|^{-\frac{1}{2}} + 2 \right)^2 O(1) + M \cdot O(|X|^{-\frac{1}{2}}) \end{aligned}$$

Similarly, we can compute  $\nabla_i (\mathbf{a}^{ij} \nabla_j \Psi)$  and estimate it by

$$\begin{aligned} (4.61) \quad \nabla_i (\mathbf{a}^{ij} \nabla_j \Psi) &= \mathbf{a}^{ij} \nabla_{ij}^2 \Psi + (\nabla_i \mathbf{a}^{ij}) (\nabla_j \Psi) \\ &= \left( \frac{3}{2} M(t + \tau) |X|^{-\frac{1}{2}} + 2 \right)^2 O(1) + \left( \frac{3}{2} M(t + \tau) |X|^{-\frac{1}{2}} + 2 \right) O(|X|^{-2}) + M \cdot O(|X|^{-\frac{1}{2}}) \end{aligned}$$

Then (4.46) follows from (4.57), (4.60) and (4.61).  $\square$

Using the above two lemmas, we can derive the following Carleman's inequality on the flow  $\{\Sigma_t\}_{-1 \leq t \leq 0}$  (with  $\Sigma_0 = \mathcal{C}$ ).

**Proposition 4.11** (Carleman's inequality). *Assume that  $\varkappa \leq 6^{-4} \lambda^3$  in (2.4) and (2.5). Then there exists  $R \geq 1$  (depending on  $\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \|F\|_{C^3(U)}, \lambda, \varkappa$ ) so that for any constants  $M \geq 1, \tau \in (0, 1]$ , and one-parameter family of  $C^2$  functions  $u_t = u(\cdot, t)$  which is compactly supported in  $\Sigma_t \setminus \bar{B}_R$  for each  $t \in [-\tau, 0]$  and is differentiable in time, there holds*

$$\begin{aligned} (4.62) \quad &\frac{\lambda^2}{9} \int_{-\tau}^0 \int_{\Sigma_t} (|\nabla_{\Sigma_t} u|^2 + u^2) G d\mathcal{H}^n dt \\ &\leq \left\{ \int_{-\tau}^0 \int_{\Sigma_t} |\mathbf{P}u|^2 G d\mathcal{H}^n dt + \frac{3}{\lambda} \int_{\Sigma_{-\tau}} |\nabla_{\Sigma_{-\tau}} u_{-\tau}|^2 G(\cdot, -\tau) d\mathcal{H}^n \right\} \end{aligned}$$

$$+\frac{1}{2}\int_{\mathcal{C}}\Psi(\cdot,0)u^2(\cdot,0)G(\cdot,0)d\mathcal{H}^n\}$$

where  $\mathcal{H}^n$  is the  $n$ -dimensional Hausdorff measure;  $\mathbf{P}$ ,  $G$  and  $\Psi$  are defined in (3.48), (4.43), (4.44), respectively.

*Proof.* Apply Lemma 4.8 to the hypersurface  $\Sigma_t$  (with  $\partial_t g = -2F(A^\#)A$ ), the differential operator  $\mathbf{P}$  and the function  $u_t$  to get

$$\begin{aligned} & \int_{\Sigma_t} \left\{ (2\Upsilon^{ij} - (\Phi - \Psi)\mathbf{a}^{ij}) \nabla_i u \nabla_j u + \frac{1}{2} (\partial_t \Psi - \nabla_i (a^{ij} \nabla_j \Psi) + (\Phi - \Psi)\Psi) u^2 \right\} G d\mathcal{H}^n \\ &= \int_{\Sigma_t} 2\mathbf{P}u \left( \partial_t u + \mathbf{a}^{ij} \nabla_i \ln G \nabla_j u + \frac{1}{2} \Psi u \right) G d\mathcal{H}^n - \int_{\Sigma_t} 2 \left( \partial_t u + \mathbf{a}^{ij} \nabla_i \ln G \nabla_j u + \frac{1}{2} \Psi u \right)^2 G d\mathcal{H}^n \\ & \quad - \partial_t \left\{ \int_{\Sigma_t} \left( \mathbf{a}^{ij} \nabla_i u \nabla_j u - \frac{1}{2} \Psi u^2 \right) G d\mathcal{H}^n \right\} \end{aligned} \quad (4.63)$$

By Cauchy-Schwarz inequality, the RHS of (4.63) is bounded from above by

$$(4.64) \quad \int_{\Sigma_t} |\mathbf{P}u|^2 G d\mathcal{H}^n dt - \partial_t \left\{ \int_{\Sigma_t} \left( \mathbf{a}^{ij} \nabla_i u \nabla_j u - \frac{1}{2} \Psi u^2 \right) G d\mathcal{H}^n \right\}$$

By Lemma 4.9 and  $R \geq 1$ , the LHS of (4.63) is bounded from below by

$$(4.65) \quad \frac{\lambda^2}{9} \int_{\Sigma_t} (|\nabla_{\Sigma_t} u|^2 + u^2) G d\mathcal{H}^n$$

Combining (4.63), (4.64), (4.65), we get

$$\begin{aligned} (4.66) \quad & \frac{\lambda^2}{9} \int_{\Sigma_t} (|\nabla_{\Sigma_t} u|^2 + u^2) G d\mathcal{H}^n \\ & \leq \int_{\Sigma_t} |\mathbf{P}u|^2 G d\mathcal{H}^n dt - \partial_t \left\{ \int_{\Sigma_t} \left( \mathbf{a}^{ij} \nabla_i u \nabla_j u - \frac{1}{2} \Psi u^2 \right) G d\mathcal{H}^n \right\} \end{aligned}$$

Integrate (4.66) in time from  $-\tau$  to 0 and then use (3.52) and  $\Psi \geq 0$  to conclude (4.62).  $\square$

Now we are ready to show that  $h$  vanishes outside a compact set. We basically follows the proof in [ESS] (which is also used in [W]).

**Theorem 4.12.** *Suppose that  $\varkappa \leq 6^{-4}\lambda^3$  in (2.4) and (2.5), then there exists  $\mathbf{R} = \mathbf{R}(\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \|F\|_{C^3(U)}, \lambda, \varkappa) \geq 1$  so that the deviation  $h(\cdot, -1)$  of  $\tilde{\Sigma}$  from  $\Sigma$  vanishes on  $\Sigma \setminus \bar{B}_{\mathbf{R}}$ . In other words,  $\tilde{\Sigma} = \Sigma$  outside the ball  $B_{\mathbf{R}}$ .*

*Proof.* Choose  $R \gg 1$  (depending on  $\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \|F\|_{C^3(U)}, \lambda$ ) so that Proposition 3.7, Proposition 3.8, Proposition 4.7, Proposition 4.11 and (3.15) hold; in particular, we may assume that for all  $X \in \Sigma_t \setminus \bar{B}_R$ ,  $t \in [-\tau, 0]$

$$(4.67) \quad |\mathbf{P}h| \leq \frac{\lambda}{6} (|\nabla_{\Sigma_t} h| + |h|)$$

$$(4.68) \quad |\nabla_{\Sigma_t} h| + |h| \leq \Lambda \exp\left(\frac{|X|^2}{\Lambda t}\right)$$

where  $\Lambda = \Lambda(n, \mathcal{C}, \|F\|_{C^3(U)}, \lambda) > 0$ ,  $\tau \equiv \min\{\alpha(n, \mathcal{C}, \|F\|_{C^3(U)}, \lambda), \frac{1}{\Lambda}\}$  (see Proposition 4.7).

For any given  $M \geq 1$  and  $\mathcal{R} \geq 4R+1$ , choose a smooth cut-off function  $\zeta = \zeta(X)$  so that

$$(4.69) \quad \chi_{B_{\mathcal{R}-1} \setminus \bar{B}_{R+1}} \leq \zeta \leq \chi_{B_{\mathcal{R}} \setminus \bar{B}_R}$$

$$|D\zeta| + |D^2\zeta| \leq 3$$

Note that  $D\zeta$  is supported in  $E = \left\{ X \in \mathbb{R}^{n+1} \mid R \leq |X| \leq R+1 \text{ or } \mathcal{R}-1 \leq |X| \leq \mathcal{R} \right\}$ .

Let  $u(\cdot, t) = \zeta h(\cdot, t)$ , then  $u(\cdot, t)$  is compactly supported in  $\Sigma_t \setminus \bar{B}_R$  for each  $t \in [-\tau, 0]$ , and we have, by (4.67), (4.68), (4.69)

$$(4.70) \quad \begin{aligned} |\mathbf{P}u| &= \left| \zeta \mathbf{P}h - h \mathbf{P}\zeta - 2\mathbf{a}^{ij} \nabla_i \zeta \nabla_j h \right| \\ &\leq \frac{\lambda}{6} (|\nabla_{\Sigma_t} u| + |u|) + C(n, \mathcal{C}, \|F\|_{C^3(U)}) (|\nabla_{\Sigma_t} h| + |h|) \chi_E \\ &\leq \frac{\lambda}{6} (|\nabla_{\Sigma_t} u| + |u|) + C(n, \mathcal{C}, \|F\|_{C^3(U)}, \lambda) \exp\left(\frac{|X|^2}{\Lambda t}\right) \chi_E \end{aligned}$$

$$(4.71) \quad u(\cdot, 0) = 0$$

By (4.70), (4.71), Proposition 4.11 and (4.68), we get

$$(4.72) \quad \begin{aligned} \frac{\lambda^2}{9} \int_{-\tau}^0 \int_{\Sigma_t} (|\nabla_{\Sigma_t} u|^2 + u^2) G d\mathcal{H}^n dt &\leq \left\{ \frac{\lambda^2}{18} \int_{-\tau}^0 \int_{\Sigma_t} (|\nabla_{\Sigma_t} u|^2 + u^2) G d\mathcal{H}^n dt \right. \\ &\quad + C(n, \mathcal{C}, \|F\|_{C^3(U)}, \lambda) \int_{-\tau}^0 \int_{\Sigma_t \cap E} \exp\left(2\frac{|X|^2}{\Lambda t}\right) G d\mathcal{H}^n dt \\ &\quad \left. + C(n, \mathcal{C}, \|F\|_{C^3(U)}, \lambda) \int_{\Sigma_{-\tau}} \exp\left(-2\frac{|X|^2}{\Lambda \tau}\right) G(\cdot, -\tau) d\mathcal{H}^n \right\} \end{aligned}$$

where  $G$  is defined in (4.43). Note that by the choice  $\tau \leq \frac{1}{\Lambda}$ , we can estimate the last two terms on the RHS of (4.72) by

$$(4.73) \quad \int_{-\tau}^0 \int_{\Sigma_t \cap E} \exp\left(2\frac{|X|^2}{\Lambda t}\right) G d\mathcal{H}^n dt \leq \int_{-\tau}^0 \int_{\Sigma_t \cap E} \exp\left(M\tau|X|^{\frac{3}{2}} - |X|^2\right) d\mathcal{H}^n dt$$

and

$$(4.74) \quad \int_{\Sigma_{-\tau}} \exp\left(-2\frac{|X|^2}{\Lambda \tau}\right) G(\cdot, -\tau) d\mathcal{H}^n \leq \int_{\Sigma_{-\tau}} \exp(-|X|^2) d\mathcal{H}^n$$

Consequently, by (4.73), (4.74) and noting that the first term on the RHS of (4.72) can be absorbed by its LHS, we get from (4.72) that

$$(4.75) \quad \begin{aligned} \frac{\lambda^2}{18} \int_{-\tau}^0 \int_{\Sigma_t} (|\nabla_{\Sigma_t} u|^2 + u^2) G d\mathcal{H}^n dt \\ \leq \left\{ C(n, \mathcal{C}, \|F\|_{C^3(U)}, \lambda) \int_{-\tau}^0 \int_{\Sigma_t \cap E} \exp\left(M\tau|X|^{\frac{3}{2}} - |X|^2\right) d\mathcal{H}^n dt \right. \\ \quad \left. + C(n, \mathcal{C}, \|F\|_{C^3(U)}, \lambda) \int_{\Sigma_{-\tau}} \exp(-|X|^2) d\mathcal{H}^n \right\} \\ \leq \left\{ C(n, \mathcal{C}, \|F\|_{C^3(U)}, \lambda) \int_{-\tau}^0 \int_{\Sigma_t \cap (B_{\mathcal{R}-1} \setminus \bar{B}_{\mathcal{R}})} \exp\left(M\tau\mathcal{R}^{\frac{3}{2}} - (\mathcal{R}-1)^2\right) d\mathcal{H}^n dt \right. \end{aligned}$$

$$\begin{aligned}
& +C(n, \mathcal{C}, \|F\|_{C^3(U)}, \lambda) \int_{-\tau}^0 \int_{\Sigma_t \cap (B_R \setminus \bar{B}_{R+1})} \exp\left(M\tau(R+1)^{\frac{3}{2}} - R^2\right) d\mathcal{H}^n dt \\
& +C(n, \mathcal{C}, \|F\|_{C^3(U)}, \lambda) \int_{\Sigma_{-\tau}} \exp(-|X|^2) d\mathcal{H}^n\}
\end{aligned}$$

The first term on the RHS of (4.75) goes away as  $\mathcal{R} \nearrow \infty$ ; the last term is bounded from above by  $C(n, \mathcal{C}, \|F\|_{C^3(U)}, \lambda)$  because of (3.5). For the LHS of (4.75), we have

$$\begin{aligned}
\frac{\lambda^2}{18} \int_{-\tau}^0 \int_{\Sigma_t} (|\nabla_{\Sigma_t} u|^2 + u^2) G d\mathcal{H}^n dt & \geq \frac{\lambda^2}{18} \int_{-\frac{\tau}{2}}^0 \int_{\Sigma_t \cap (B_{\mathcal{R}-1} \setminus \bar{B}_{4R})} u^2 G d\mathcal{H}^n dt \\
& \geq \frac{\lambda^2}{18} \exp\left(4M\tau R^{\frac{3}{2}}\right) \int_{-\frac{\tau}{2}}^0 \int_{\Sigma_t \cap (B_{\mathcal{R}-1} \setminus \bar{B}_{4R})} h^2 d\mathcal{H}^n dt
\end{aligned}$$

Therefore, let  $\mathcal{R} \nearrow \infty$  in (4.75), we arrive at

$$\begin{aligned}
(4.76) \quad & \int_{-\frac{\tau}{2}}^0 \int_{\Sigma_t \setminus \bar{B}_{4R}} h^2 d\mathcal{H}^n dt \\
& \leq \exp\left(-4M\tau R^{\frac{3}{2}}\right) C(n, \mathcal{C}, \|F\|_{C^3(U)}, \lambda) \left\{ \exp\left(2\sqrt{2}M\tau R^{\frac{3}{2}}\right) + 1 \right\}
\end{aligned}$$

Let  $M \nearrow \infty$  in (4.76), we get  $h_t = h(\cdot, t)$  vanishes on  $\Sigma_t \setminus \bar{B}_{4R}$  for  $t \in [-\frac{\tau}{2}, 0]$ , and hence  $\tilde{\Sigma}_{-\frac{\tau}{2}} = \sqrt{\frac{\tau}{2}} \tilde{\Sigma}$  coincides with  $\Sigma_{-\frac{\tau}{2}} = \sqrt{\frac{\tau}{2}} \Sigma$  outside  $B_{4R}$ , which in turn shows that  $\tilde{\Sigma}$  coincide with  $\Sigma$  outside the ball of radius  $R = \frac{4R}{\sqrt{\tau/2}}$ .  $\square$

By the previous theorem and the “unique continuation principle” in Proposition 4.5 (see Remark 4.6), we have the following characterization of the overlap region of  $\Sigma$  and  $\tilde{\Sigma}$ .

**Theorem 4.13.** *Under the same hypothesis of Theorem 4.12, let*

$$\Sigma^0 = \left\{ X \in \Sigma \cap \tilde{\Sigma} \mid \Sigma \text{ coincides with } \tilde{\Sigma} \text{ in a neighborhood of } X \right\}$$

then  $\Sigma^0$  is a nonempty hypersurface and  $\partial\Sigma^0 \subseteq (\partial\Sigma \cup \partial\tilde{\Sigma})$ .

*Proof.* Note that  $\Sigma^0$  is a nonempty hypersurface follows from Theorem 4.12.

Suppose that  $\partial\Sigma^0 \not\subseteq (\partial\Sigma \cup \partial\tilde{\Sigma})$ , then pick  $\hat{X} \in \partial\Sigma^0 \setminus (\partial\Sigma \cup \partial\tilde{\Sigma})$  and choose a sequence  $\{\hat{X}_m \in \Sigma^0\}$  converging to  $\hat{X}$ . Note that  $N(\hat{X}) = \tilde{N}(\hat{X})$  since  $N(\hat{X}_m) = \tilde{N}(\hat{X}_m)$  for all  $m \in \mathbb{N}$ , where  $N, \tilde{N}$  are the unit-normal of  $\Sigma$  and  $\tilde{\Sigma}$ , respectively. Thus, near  $\hat{X}$ ,  $\Sigma$  and  $\tilde{\Sigma}$  can be regraded as graphes of  $u$  and  $\tilde{u}$ , respectively, over  $B_{\hat{\rho}}^n \subset T_{\hat{X}}\Sigma = T_{\hat{X}}\tilde{\Sigma}$  for some  $\hat{\rho} \in (0, 1)$ . That is,  $\Sigma$  and  $\tilde{\Sigma}$  can be respectively parametrized by

$$X = X(x) \equiv \hat{X} + (x, u(x)), \quad \tilde{X} = \tilde{X}(x) \equiv \hat{X} + (x, \tilde{u}(x)) \quad \text{for } x \in B_{\hat{\rho}}^n$$

in which we assume that  $N(\hat{X}) = \tilde{N}(\hat{X}) = (0, 1)$  for ease of notation. Note also that  $A_i^j(0) = \tilde{A}_i^j(0)$  since  $A_i^j(x_m) = \tilde{A}_i^j(x_m)$  for all  $m \in \mathbb{N}$ , where  $x_m$  is the

coordinates of  $\hat{X}_m$  (i.e.  $X(x_m) = \hat{X}_m = \tilde{X}(x_m)$ ) and  
(4.77)

$$A^\#(x) \sim A_i^j(x) = \partial_i \left( \frac{\partial_j \mathbf{u}(x)}{\sqrt{1 + |\partial_x \mathbf{u}|^2}} \right), \quad \tilde{A}^\#(x) \sim \tilde{A}_i^j(x) = \partial_i \left( \frac{\partial_j \tilde{\mathbf{u}}(x)}{\sqrt{1 + |\partial_x \tilde{\mathbf{u}}|^2}} \right)$$

are the shape operators of  $\Sigma$  and  $\tilde{\Sigma}$ , respectively. As a result, we may assume (by choosing  $\boldsymbol{\varrho}$  small if necessary) that  $\tilde{A}_i^j(x)$  is so close to  $A_i^j(x)$  that the set

$$\mathfrak{U} = \left\{ (1 - \theta) A_i^j(x) + \theta \tilde{A}_i^j(x) \mid x \in B_{\boldsymbol{\varrho}}^n, \theta \in [0, 1] \right\}$$

is a bounded subset of  $\Omega$  and there holds

$$\bar{\lambda} \leq \frac{\partial F}{\partial S_i^j} \left( (1 - \theta) A^\#(x) + \theta \tilde{A}^\#(x) \right) \leq \frac{1}{\bar{\lambda}}$$

for some  $\bar{\lambda} \in (0, 1]$ .

From the  $F$  shrinker equation in Definition 2.4, we get

$$(4.78) \quad \sqrt{1 + |\partial_x \mathbf{u}|^2} F(A_i^j(x)) + \frac{1}{2} (\mathbf{u} - x \cdot \partial_x \mathbf{u}) = 0, \quad \sqrt{1 + |\partial_x \tilde{\mathbf{u}}|^2} F(\tilde{A}_i^j(x)) + \frac{1}{2} (\tilde{\mathbf{u}} - x \cdot \partial_x \tilde{\mathbf{u}}) = 0$$

Subtracting (4.78) and using (4.77), we then get an equation for  $\mathbf{v} = \tilde{\mathbf{u}} - \mathbf{u}$ :

$$(4.79) \quad \mathfrak{a}^{ij} \partial_{ij}^2 \mathbf{v} + \mathfrak{b}^j \partial_j \mathbf{v} + \frac{1}{2} \mathbf{v} = 0$$

where

$$(4.80) \quad \mathfrak{a}^{ij}(x) = \int_0^1 \left\{ \frac{\partial F}{\partial S_i^j} \left( (1 - \theta) A^\#(x) + \theta \tilde{A}^\#(x) \right) - \frac{\partial F}{\partial S_i^k} \left( (1 - \theta) A^\#(x) + \theta \tilde{A}^\#(x) \right) \frac{\partial_k \mathbf{u}_\theta \partial_j \mathbf{u}_\theta}{1 + |\partial_x \mathbf{u}_\theta|^2} \right\} d\theta$$

$$(4.81) \quad \mathfrak{b}^j(x) = \left\{ - \int_0^1 \frac{\partial F}{\partial S_i^j} \left( (1 - \theta) A^\#(x) + \theta \tilde{A}^\#(x) \right) \frac{\partial_k \mathbf{u}_\theta \partial_{ik}^2 \mathbf{u}_\theta}{1 + |\partial_x \mathbf{u}_\theta|^2} d\theta \right. \\ \left. - \int_0^1 \frac{\partial F}{\partial S_i^k} \left( (1 - \theta) A^\#(x) + \theta \tilde{A}^\#(x) \right) \frac{\partial_j \mathbf{u}_\theta \partial_{ik}^2 \mathbf{u}_\theta + \partial_k \mathbf{u}_\theta \partial_{ij}^2 \mathbf{u}_\theta}{1 + |\partial_x \mathbf{u}_\theta|^2} d\theta \right. \\ \left. + 3 \int_0^1 \frac{\partial F}{\partial S_i^k} \left( (1 - \theta) A^\#(x) + \theta \tilde{A}^\#(x) \right) \frac{\partial_j \mathbf{u}_\theta \partial_k \mathbf{u}_\theta \partial_l \mathbf{u}_\theta \partial_{il}^2 \mathbf{u}_\theta}{(1 + |\partial_x \mathbf{u}_\theta|^2)^{\frac{3}{2}}} d\theta \right. \\ \left. + \int_0^1 F \left( (1 - \theta) A^\#(x) + \theta \tilde{A}^\#(x) \right) \frac{\partial_j \mathbf{u}_\theta}{\sqrt{1 + |\partial_x \mathbf{u}_\theta|^2}} d\theta - \frac{1}{2} x_j \right\}$$

and

$$\mathbf{u}_\theta = (1 - \theta) \mathbf{u} + \theta \tilde{\mathbf{u}}$$

Note that (4.79) is equivalent to the following divergence form equation:

$$(4.82) \quad - \partial_i \left( \frac{\mathfrak{a}^{ij} + \mathfrak{a}^{ji}}{2} \partial_j \mathbf{v} \right) = \left( - \partial_i \left( \frac{\mathfrak{a}^{ij} + \mathfrak{a}^{ji}}{2} \right) + \mathfrak{b}^j \right) \partial_j \mathbf{v} + \frac{1}{2} \mathbf{v}$$

And by (4.80), (4.81) and (4.77), we have the following estimates for the coefficients of (4.82):

$$(4.83) \quad \frac{\bar{\lambda}}{1 + \|\partial_x \mathbf{u}_\theta\|_{L^\infty(B_{\boldsymbol{\varrho}}^n)}^2} \leq \frac{\mathfrak{a}^{ij} + \mathfrak{a}^{ji}}{2} \leq C \left( \|F\|_{C^1(\mathfrak{U})}, \|\mathbf{u}\|_{C^2(B_{\boldsymbol{\varrho}}^n)} \right)$$

$$(4.84) \quad |\partial_x \mathbf{a}^{ij}| + |\mathbf{b}^j| \leq C \left( \|F\|_{C^2(\mathcal{U})}, \|\mathbf{u}\|_{C^3(B_{\frac{1}{2}}^n)} \right)$$

On the other hand, since  $\hat{X}_m \in \Sigma^0$  and  $\hat{X}_m \rightarrow \hat{X}$  as  $m \nearrow \infty$ ,  $\mathbf{v}$  is vanishing at each neighborhood of  $x_m$  and  $x_m \rightarrow 0$  as  $m \nearrow \infty$ . Thus, by Proposition 4.5 and Remark 4.6,  $\mathbf{v}$  vanishes on  $B^n(x_m, \frac{1}{4}(\varrho - |x_m|))$  for all  $m \in \mathbb{N}$ , which implies that  $\mathbf{v}$  vanishes on  $B^n(0, \frac{1}{4}\varrho)$ . In other words,  $\Sigma$  coincides with  $\tilde{\Sigma}$  in a neighborhood of  $\hat{X}$ , which contradicts with  $\hat{X} \in \partial\Sigma^0$ .  $\square$

Lastly, we give an estimate of  $\varkappa$  (defined in (2.5)) in the rotationally symmetric case to conclude this section. From now on, we assume that the cone  $\mathcal{C}$  (in Definition 2.1) is rotationally symmetric, say

$$\mathcal{C} = \left\{ (\sigma s \nu, s) \mid \nu \in \mathbf{S}^{n-1}, s \in \mathbb{R}_+ \right\}$$

for some constant  $\sigma > 0$ , where  $\mathbf{S}^{n-1}$  is the unit-sphere in  $\mathbb{R}^n$ . To derive the estimate, we have to first compute the covariant derivatives of the second fundamental form of  $\mathcal{C}$ .

**Lemma 4.14.** *At each point  $X_{\mathcal{C}} = (\sigma s \nu, s) \in \mathcal{C}$  (where  $\nu \in \mathbf{S}^{n-1}$ ,  $s > 0$ ), pick an orthonormal basis  $\{e_1^{\mathcal{C}}, \dots, e_n^{\mathcal{C}}\}$  for  $T_{X_{\mathcal{C}}}\mathcal{C}$  so that  $e_n^{\mathcal{C}} = \frac{(\sigma\nu, 1)}{\sqrt{1+\sigma^2}}$ , then we have*

$$(4.85) \quad A_{\mathcal{C}}(e_i^{\mathcal{C}}, e_j^{\mathcal{C}}) = \kappa_i^{\mathcal{C}} \delta_{ij}, \quad \text{with } \kappa_1^{\mathcal{C}} = \dots = \kappa_{n-1}^{\mathcal{C}} = \frac{1}{\sigma|X_{\mathcal{C}}|}, \kappa_n^{\mathcal{C}} = 0$$

$$(4.86) \quad \nabla_{\mathcal{C}} A_{\mathcal{C}}(e_i^{\mathcal{C}}, e_j^{\mathcal{C}}, e_n^{\mathcal{C}}) = \frac{-1}{\sigma|X_{\mathcal{C}}|^2} \delta_{ij} = -\frac{\kappa_i^{\mathcal{C}}}{|X_{\mathcal{C}}|} \delta_{ij}, \quad \forall i, j \neq n$$

$$(4.87) \quad \nabla_{\mathcal{C}} A_{\mathcal{C}}(e_i^{\mathcal{C}}, e_j^{\mathcal{C}}, e_k^{\mathcal{C}}) = \nabla_{\mathcal{C}} A_{\mathcal{C}}(e_i^{\mathcal{C}}, e_n^{\mathcal{C}}, e_n^{\mathcal{C}}) = \nabla_{\mathcal{C}} A_{\mathcal{C}}(e_n^{\mathcal{C}}, e_n^{\mathcal{C}}, e_n^{\mathcal{C}}) = 0 \quad \forall i, j, k \neq n$$

where  $A_{\mathcal{C}}$  is the second fundamental form of  $\mathcal{C}$  and  $\nabla_{\mathcal{C}} A_{\mathcal{C}}$  is its covariant derivative. Note that  $A_{\mathcal{C}}$  and  $\nabla_{\mathcal{C}} A_{\mathcal{C}}$  are totally symmetric tensors (by Codazzi equation).

*Proof.* Let's parametrize  $\mathcal{C}$  by

$$X_{\mathcal{C}} = (\sigma s \nu, s) \quad \text{for } \nu \in \mathbf{S}^{n-1}, s \in \mathbb{R}_+$$

and take an orthonormal local frame  $\{e_1^{\mathcal{C}}, \dots, e_n^{\mathcal{C}}\}$  of  $\mathcal{C}$  so that

$$(4.88) \quad e_n^{\mathcal{C}} = \frac{\partial_s X_{\mathcal{C}}}{|\partial_s X_{\mathcal{C}}|} = \frac{(\sigma\nu, 1)}{\sqrt{1+\sigma^2}}$$

By a simple calculation, the principal curvatures of the cone  $\mathcal{C}$  are given by

$$(4.89) \quad \kappa_1^{\mathcal{C}} = \dots = \kappa_{n-1}^{\mathcal{C}} = \frac{1}{\sigma s \sqrt{1+\sigma^2}} = \frac{1}{\sigma|X_{\mathcal{C}}|}, \quad \kappa_n^{\mathcal{C}} = 0$$

Since  $\{e_1^{\mathcal{C}}, \dots, e_n^{\mathcal{C}}\}$  forms a principal basis at each point, so by (4.89) we have

$$(4.90) \quad A_{ii}^{\mathcal{C}} = \kappa_i^{\mathcal{C}} = \frac{1}{\sigma s \sqrt{1+\sigma^2}} = \frac{1}{\sigma|X_{\mathcal{C}}|} \quad \text{whenever } i \neq n$$

$$A_{ij}^{\mathcal{C}} = 0 = A_{nn}^{\mathcal{C}} \quad \text{whenever } i \neq j$$

where  $A_{ij}^{\mathcal{C}} \equiv A_{\mathcal{C}}(e_i^{\mathcal{C}}, e_j^{\mathcal{C}})$ .

By the orthonormality of  $\{e_1^C, \dots, e_n^C\}$  and the product rule, the Christoffel symbols  ${}^C\Gamma_{ij}^k \equiv (D_{e_i^C} e_j^C) \cdot e_k^C$  satisfy

$$(4.91) \quad {}^C\Gamma_{ki}^j = (D_{e_k^C} e_i^C) \cdot e_j^C = - (D_{e_k^C} e_j^C) \cdot e_i^C = -{}^C\Gamma_{kj}^i$$

Thus, from (4.90) and (4.91), we deduce that whenever  $i, j \neq n$  or  $i = j = n$ , there holds

$$(4.92) \quad \nabla_k^C A_{ij}^C = D_{e_k^C} (A_{ij}^C) - {}^C\Gamma_{ki}^j A_{jj}^C - {}^C\Gamma_{kj}^i A_{ii}^C = D_{e_k^C} (A_{ij}^C)$$

By (4.92), (4.90) and (4.88), we get

$$\begin{aligned} \nabla_n^C A_{ij}^C &= D_{e_n^C} (\kappa_i^C \delta_{ij}) = \frac{1}{\sqrt{1+\sigma^2}} \partial_s \left( \frac{1}{\sigma s \sqrt{1+\sigma^2}} \right) \delta_{ij} \\ &= \frac{-1}{\sigma(1+\sigma^2)s^2} \delta_{ij} = \frac{-1}{\sigma |X_C|^2} \delta_{ij} \quad \text{if } i, j \neq n \end{aligned}$$

which verifies (4.86).

By (4.92), (4.90) and noting that  $|X_C|$  is invariant along  $e_k^C$  for  $k \neq n$ , we get

$$(4.93) \quad \nabla_k^C A_{ij}^C = D_{e_k^C} (\kappa_i^C \delta_{ij}) = D_{e_k^C} \left( \frac{1}{\sigma |X_C|} \right) \delta_{ij} = 0 \quad \text{if } i, j, k \neq n$$

From (4.92) and (4.90), we have

$$(4.94) \quad \nabla_i^C A_{nn}^C = D_{e_i^C} (A_{nn}^C) = 0 \quad \forall i$$

Then (4.87) follows from (4.93) and (4.94).  $\square$

Combining (2.1), (2.2), (2.3) with Lemma 4.14, we get the following:

**Proposition 4.15.** *The constant  $\varkappa$  defined in (2.5) can be estimated by*

$$(4.95) \quad \varkappa \leq C(n) \left( \left| \partial^2 f \left( \vec{1}, 0 \right) \right| + \left| \partial_1 f \left( \vec{1}, 0 \right) - \partial_n f \left( \vec{1}, 0 \right) \right| \right)$$

*Note that here we assume that  $\mathcal{C}$  is rotationally symmetric.*

*Proof.* At each point  $X_C \in \mathcal{C}$ , take an orthonormal basis  $\{e_1^C, \dots, e_n^C\}$  for  $T_{X_C} \mathcal{C}$  so that  $e_n^C = \frac{(\sigma\nu, 1)}{\sqrt{1+\sigma^2}}$ . Then by (2.2), (2.3), Lemma 4.14 and the homogeneity of the derivatives of  $f$ , we get

$$\begin{aligned} \left| \frac{\partial^2 F}{\partial S_i^j \partial S_k^l} (A_C^\#) \right| &\leq \left( \left| \partial^2 f (\kappa_1^C, \dots, \kappa_n^C) \right| + \left| \frac{\partial_1 f (\kappa_1^C, \dots, \kappa_n^C) - \partial_n f (\kappa_1^C, \dots, \kappa_n^C)}{\kappa_1^C - \kappa_n^C} \right| \right) \\ &= \frac{1}{\kappa_1^C} \left( \left| \partial^2 f \left( \vec{1}, 0 \right) \right| + \left| \partial_1 f \left( \vec{1}, 0 \right) - \partial_k f \left( \vec{1}, 0 \right) \right| \right) \end{aligned}$$

which implies that

$$\begin{aligned} &|X_C| \left| \sum_{k,l} \frac{\partial^2 F}{\partial S_i^j \partial S_k^l} (A_C^\#) \left( \nabla_C A_C^\# \right)_k^l \right| \\ &\leq |X_C| \frac{C(n)}{\kappa_1^C} \left( \left| \partial^2 f \left( \vec{1}, 0 \right) \right| + \left| \partial_1 f \left( \vec{1}, 0 \right) - \partial_k f \left( \vec{1}, 0 \right) \right| \right) \frac{\kappa_1^C}{|X_C|} \\ &= C(n) \left( \left| \partial^2 f \left( \vec{1}, 0 \right) \right| + \left| \partial_1 f \left( \vec{1}, 0 \right) - \partial_k f \left( \vec{1}, 0 \right) \right| \right) \end{aligned}$$



Therefore,

$$\begin{aligned} \varkappa &= \sup_{X_C \in \mathcal{C} \cap \left(B_3 \setminus \bar{B}_{\frac{1}{3}}\right)} \left| \sum_{k,l} \frac{\partial^2 F}{\partial S_i^j \partial S_k^l} \left(A_C^\# \right) \left(\nabla_C A_C^\# \right)_k^l \right| \\ &\leq C(n) \left( \left| \partial^2 f \left( \vec{1}, 0 \right) \right| + \left| \partial_1 f \left( \vec{1}, 0 \right) - \partial_k f \left( \vec{1}, 0 \right) \right| \right). \end{aligned}$$

□

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